Pearson & Ministry of Education and Technical Education

est EGYPTIAN SCHOLASTIC TEST

EST II

MATH LEVEL 1

LEVEL UP

2025 EDITION

STRATEGIES

EST II

EST II – Math Level 1

Numerations and Operations

Algebra and Functions

Plane geometry

Solid geometry

Trignometry

Data analysis, Statistics and Probability

Numbers and Operations

Numbers and Operations

1 Counting

VENN DIAGRAMS

Counting problems usually begin with the phrase "How many . . ." or the phrase "In how many ways . . ." Illustrating counting techniques by example is best.

EXAMPLE 1

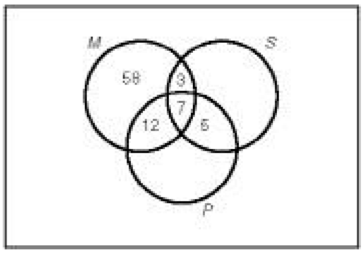
> **A certain sports club has 50 members. Of these, 35 golf, 30 hunt, and 18 do both. How many club members do neither?**

Add 35 and 30, then subtract the 18 that were counted twice. This makes 47 who golf, hunt, or do both. Therefore, only 3 (50 – 47) do neither.

EXAMPLE 2

> **Among the seniors at a small high school, 80 take math, 41 take Spanish, and 54 take physics. Ten seniors take math and Spanish; 19 take math and physics; and 12 take physics and Spanish. Seven seniors take all three. How many seniors take math but not Spanish or physics?**

A Venn diagram will help you sort out this complicated-sounding problem.



Start with the 7 who take all three courses. Since 10 take math and Spanish, this leaves 3 who take math and Spanish but not physics. Use similar reasoning to see that 5 take physics and Spanish but not math, and 12 take math and physics but not Spanish.

Finally, using the totals for how many students take each course, we can conclude that 58 (80 – 3 – 7 – 12) take math but not physics or Spanish.

EXERCISE

1. There are 50 people in a room. Twenty-eight are male, and 32 are under the age of 30. Twelve are males under the age of 30. How many women over the age of 30 are in the group?
   (A) 2
   (B) 3
   (C) 4
   (D) 5
   (E) 6

MULTIPLICATION RULE

Many other counting problems use the multiplication principle.

EXAMPLE 1

Suppose you have 5 shirts, 4 pairs of pants, and 9 ties. How many outfits can be made consisting of a shirt, a pair of pants, and a tie?

For each of the 5 shirts, you can wear 4 pairs of pants, so there are 5 · 4 = 20 shirt-pants combinations. For each of these 20 shirt-pants combinations, there are 9 ties, so there are 20 · 9 = 180 shirt-pants-tie combinations.

EXAMPLE 2

> **Six very good friends decide they will have lunch together every day. In how many different ways can they line up in the lunch line?**

Any one of the 6 could be first in line. For each person who is first, there are 5 who could be second. This means there are 30 ($6 \cdot 5$) ways of choosing the first two people. For each of these 30 ways, there are 4 ways of choosing the third person. This makes 120 ($30 \cdot 4$) ways of choosing the first 3 people. Continuing in this fashion, there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ ways these 6 friends can stand in the cafeteria line. (This means that if they all have perfect attendance for 4 years of high school, they could stand in line in a different order every day, because $720 = 4 \cdot 180$.)

EXAMPLE 3

> **The math team at East High has 20 members. They want to choose a president, vice president, and treasurer. In how many ways can this be done?**

Any one of the 20 members could be president. For each choice, there are 19 who could be vice president. For each of these 380 ($20 \cdot 19$) ways of choosing a president and a vice president, there are 18 choices for treasurer. Therefore, there are $20 \cdot 19 \cdot 18 = 6840$ ways of choosing these three club officers.

EXAMPLE 4

> **The student council at West High has 20 members. They want to select a committee of 3 to work with the school administration on policy matters affecting students directly. How many committees of 3 students are possible?**

This problem is similar to example 3, so we will start with the fact that if they were electing 3 officers, the student council would be able to do this in 6840 ways. However, it does not matter whether member A is president, B is vice president, and C is treasurer or some other arrangement, as long as all 3 are on the committee. Therefore, we can divide 6840 by the number of ways the 3 students selected could be president, vice president, and treasurer. This latter number is $3 \cdot 2 \cdot 1 = 6$, so there are 1140 ($6840 \div 6$) committees of 3.

EXERCISES

1. M & M plain candies come in six colors: brown, green, orange, red, tan, and yellow. Assume there are at least 3 of each color. If you pick three candies from a bag, how many color possibilities are there?
   (A) 18
   (B) 20
   (C) 120
   (D) 216
   (E) 729
2. A code consists of two letters of the alphabet followed by 5 digits. How many such codes are possible?
   (A) 7
   (B) 10
   (C) 128
   (D) 20,000
   (E) 67,600,000
3. A salad bar has 7 ingredients, excluding the dressing. How many different salads are possible where two salads are different if they don't include identical ingredients?
   (A) 7
   (B) 14
   (C) 128
   (D) 5,040
   (E) 823,543

FACTORIAL, PERMUTATIONS, COMBINATIONS

Counting problems like the ones in the last three examples occur frequently enough that they have special designations.

Ordering nObjects (Factorial)

The second example of the multiplication rule asked for the number of ways 6 friends could stand in line. By using the multiplication principle, we found that there were $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways. A special notation for this product is 6! (6 factorial). In general, the number of ways objects can be ordered is n!

Ordering r of n Objects (Permutations)

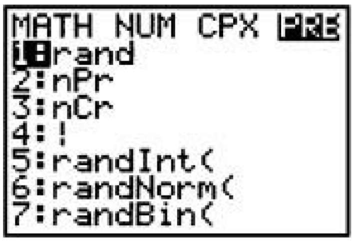
The third example of the multiplication rule asked for the number of ways you could choose a first (president), second (vice president), and third person (secretary) out of 20 people (r = 3, n = 20). The answer is 20 · 19 · 18, or $\frac{20!}{17!} = \frac{20!}{(20-3)!}$. In general, there are permutations of r objects of n. This appears as $_nP_r$ in the calculator menu.

Choosing r of n Objects (Combinations)

In the fourth example of the multiplication rule, we were interested in choosing a committee of 3 where there was no distinction among committee members. Our approach was first to compute the number of ways of choosing officers and then dividing out the number of ways the three officers could hold the different offices. This led to the computation $\frac{20!}{17!3!} = \frac{20!}{(20-3)!3!}$. In general, the number of ways of choosing r of n objects is $\frac{n!}{(n-r)!r!}$. This quantity appears on the calculator menu as $_nC_r$. However, there is a special notation for combinations:

$_nC_r = \binom{n}{r} =$ the number of ways r objects can be chosen from n

Calculator commands for all three of these functions are in the MATH/PRB menu.



These 3 commands can also be found on scientific calculators.

EXERCISES

1. How many 3-person committees can be selected from a fraternity with 25 members?
   (A) 15,625
   (B) 13,800
   (C) 2,300

(D) 75
(E) 8

2. A basketball team has 5 centers, 9 guards, and 13 forwards. Of these, 1 center, 2 guards, and 2 forwards start a game. How many possible starting teams can a coach put on the floor?
(A) 56,160
(B) 14,040
(C) 585
(D) 197
(E) 27

3. Five boys and 6 girls would like to serve on the homecoming court, which will consist of 2 boys and 2 girls. How many different homecoming courts are possible?
(A) 30
(B) 61
(C) 150
(D) 900
(E) 2048

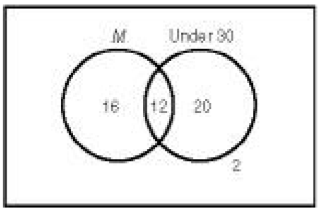
4. In a plane there are 8 points, no three of which are collinear. How many lines do the points determine?
(A) 7
(B) 16
(C) 28
(D) 36
(E) 64

5. If $\binom{6}{x} = \binom{4}{x}$, then x=
(A) 0
(B) 1
(C) 4
(D) 5
(E) 10

Answers and Explanations

Venn Diagrams

1. **(A)** A Venn diagram will help you solve this problem.



The two circles represent males and people who are at most 30 years of age, respectively. The part of the rectangle outside both circles represents people who are in neither category, i.e., females over the age of 30. First fill in the 12 males who are less than or equal to 30 years of age in the intersection of the circles. Since there are 28 males altogether, 16 are male and over 30. Since there are 32 people age 30 or less, there are 20 women that age. Add these together to get 48 people. Since there are 50 in the group, 2 must be women over 30.

Multiplication Rule

1. **(D)** There are 6 choices of color for each of the three candies selected. Therefore, there are $6 \times 6 \times 6 = 216$ color possibilities altogether.

2. **(E)** The multiplication rule applies. There are $(26)(26)(10)^5 = 67{,}600{,}000$ possible codes.

3. **(C)** You can either include or exclude each of the seven ingredients in your salad, which means there are 2 choices for each ingredient. According to the multiplication rule, there are $2^7 = 128$ ways of making these yes-no choices.

Factorial, Permutations, Combinations

1. **(C)** This is the number of ways 3 objects can be chosen from 25, or $\binom{25}{3}$ = 25_nC_r3 = 2,300..

2. **(B)** There are $\binom{5}{1} = 5$ ways of choosing the one center, $\binom{9}{2} = 36$ ways of choosing the two guards, and $\binom{13}{2} = 78$ ways of choosing the two forwards. Therefore, there are $5 \times 36 \times 78 = 14{,}040$ possible starting teams.

3. **(C)** There are $\binom{5}{2} = 10$ ways of choosing 2 boys out of 5 and $\binom{6}{2} = 15$ ways of choosing 2 girls out of 6. Therefore, there are $10 \times 15 = 150$ ways of

choosing the homecoming court.


4. **(C)** Since no three points are collinear, every pair of points determines a distinct line. There are $\binom{8}{2} = 28$ such lines.

5. **(A)** $\binom{n}{0} = 1$ for any n.

Matrices

ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION

A matrix is a rectangular array of numbers. The size of a matrix is r by c, whereri
are
matric
equal.

EXAN

$$\begin{pmatrix} 5 & -3 \\ x & 4 \end{pmatrix} = \begin{pmatrix} y-3 & -3 \\ 2x+2 & 4 \end{pmatrix}.$$

These two matrices are equal if y– 3 = 5 and x= 2x+ 2. Therefore, y= 8 and x
2.

= 1, the matrix is called a row matrix. If c= 1, the matrix is called a column
x. If r = c, the matrix is called a square matrix. The numbers from the
eft corner to the bottom right corner of a square matrix form the main
al.
ar multiplication takes place when each number in a matrix is multiplied
nstant. If two matrices are the same size, they can be added or subtracted
ng or subtracting corresponding entries.

EXAMPLE 2

Simplify: $3\begin{pmatrix} -2 & 3 \\ 1 & 5 \\ -4 & 3 \end{pmatrix} - 2\begin{pmatrix} 3 & -1 \\ 2 & 1 \\ -4 & 6 \end{pmatrix}$

$$3\begin{pmatrix} -2 & 3 \\ 1 & 5 \\ -4 & 3 \end{pmatrix} - 2\begin{pmatrix} 3 & -1 \\ 2 & 1 \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} -6 & 9 \\ 3 & 15 \\ -12 & 9 \end{pmatrix} - \begin{pmatrix} 6 & -2 \\ 4 & 2 \\ -8 & 12 \end{pmatrix} = \begin{pmatrix} -12 & 11 \\ -1 & 13 \\ -4 & -3 \end{pmatrix}$$

EXAMPLE 3

Solve the matrix equation: $2x + \begin{pmatrix} -3 & 2 & 6 \\ 5 & -1 & 0 \\ 3 & -6 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -8 & -4 \\ 1 & 3 & 10 \\ -5 & 8 & 0 \end{pmatrix}$

$$2x = \begin{pmatrix} 5 & -8 & -4 \\ 1 & 3 & 10 \\ -5 & 8 & 0 \end{pmatrix} - \begin{pmatrix} -3 & 2 & 6 \\ 5 & -1 & 0 \\ 3 & -6 & -2 \end{pmatrix} = \begin{pmatrix} 8 & -10 & -10 \\ -4 & 4 & 10 \\ -8 & 14 & 2 \end{pmatrix}$$

$$x = \begin{pmatrix} 4 & -5 & -5 \\ -2 & 2 & 5 \\ -4 & 7 & 1 \end{pmatrix}$$

EXERCISES

1. $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 11 & 5 \\ -6 & 12 \end{pmatrix} = K\begin{pmatrix} 3 & 2 \\ J & M \end{pmatrix}$. Find the value of K+ J+ M
   (A) 2
   (B) 4
   (C) 6
   (D) 7
   (E) 8

_ Evaluate x and y if $\begin{pmatrix} x & 2 \\ -3 & y \end{pmatrix} = 2\begin{pmatrix} x^2 & 1 \\ -\frac{3}{2} & 3y-5 \end{pmatrix}$.
   (A) x= 0; y= 2

(B) x= 1; y= 2

(C) x= -1, 1; $y = \frac{5}{3}$

(D) $x = -\frac{1}{2}, \frac{1}{2}; y = \frac{5}{6}$

(E) x= 0, $\frac{1}{2}$

_ Solve for x: $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 3 \end{pmatrix} - x = \begin{pmatrix} 5 & 1 & 8 \\ -6 & 0 & 5 \end{pmatrix}.$

(A) $\begin{pmatrix} 4 & 1 & -11 \\ -8 & 1 & -2 \end{pmatrix}$

(B) $\begin{pmatrix} -4 & -1 & 11 \\ 8 & -1 & 2 \end{pmatrix}$

(C) $\begin{pmatrix} -5 & -2 & 24 \\ 12 & 0 & -15 \end{pmatrix}$

(D) $\begin{pmatrix} 5 & 2 & -24 \\ -12 & 0 & 15 \end{pmatrix}$

(E) $\begin{pmatrix} 6 & 3 & 5 \\ -4 & 1 & 8 \end{pmatrix}$

MATRIX MULTIPLICATION

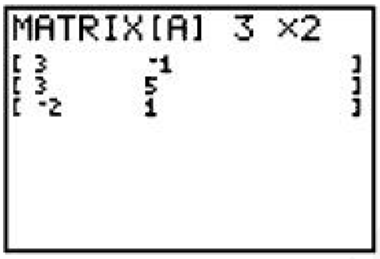
Matrix multiplication takes place when two matrices, A and B, are multiplied to orm a new matrix, AB. Matrix multiplication is possible only under certain nditions. Suppose A is r_1 by c_1 and B is r_2 by c_2. If $c_1 = r_2$, then AB is defined ize r_1 by c_2. The entry x_{ij} of AB is the ith row of A times the jth column are square matrices, BA is also defined but not generally equal

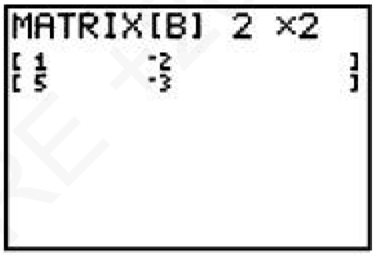
: **AB =** $\begin{pmatrix} 3 & -1 \\ 3 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 5 & -3 \end{pmatrix}$**. A is 3 by 2, and B is 2 by 2.**

he product matrix is 3 by 2, with entries x_{ij} calculated as follows: $x_{11} = (3)(1) + (-1$

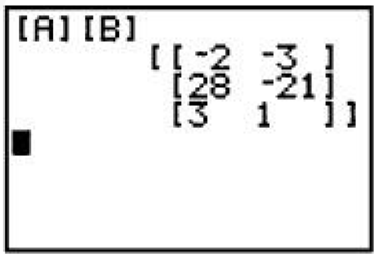
$\begin{pmatrix} 3 & -1 \\ 3 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 5 & -3 \end{pmatrix}=\begin{pmatrix} -2 & -3 \\ 28 & -21 \\ 3 & 1 \end{pmatrix}$. Note that BA is not defined.

Matrix calculations can be done on a graphing calculator. To define a matrix, enter 2nd MATRIX, highlight and enter EDIT, enter the number of rows followed by the number of columns, and finally enter the entries. The figure below shows the result of these steps for matrices A and B of Example 1.





To find the product, enter 2nd MATRIX/NAMES/[A], which returns [A] to the home screen. Also enter 2nd MATRIX/NAMES/[B], which returns [B] to the home screen. Hit ENTER again to get the product.



Square matrices of the same size can always be multiplied. However, matrix multiplication is not commutative.

EXAMPLE 21

$A=\begin{pmatrix} -2 & 5 \\ 1 & 3 \end{pmatrix}$ **and** $B=\begin{pmatrix} 6 & 0 \\ 3 & -5 \end{pmatrix}$, **evaluate AB and BA.**

$AB = \begin{pmatrix} -2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 3 & -25 \\ 15 & -15 \end{pmatrix}$, while $BA = \begin{pmatrix} 6 & 0 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -12 & 30 \\ -11 & 0 \end{pmatrix}$

EXERCISES

Use A= (-2 -1 5 9) and B= $\begin{pmatrix} 0 & -5 \\ 3 & -2 \\ 4 & 0 \\ -6 & 1 \end{pmatrix}$ for questions 1 and 2.

1. The product AB =

(A) $\begin{pmatrix} 0 & 10 \\ -3 & 2 \\ 20 & 0 \\ -54 & 9 \end{pmatrix}$

(B) $\begin{pmatrix} -37 \\ 21 \end{pmatrix}$

(C) $\begin{pmatrix} 10 \\ -2 \\ 20 \\ -45 \end{pmatrix}$

(D) $\begin{pmatrix} 0 \\ 6 \\ 0 \\ -54 \end{pmatrix}$

(E) product is not defined

2. The first row, second column of the product $\begin{pmatrix} x & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 5 & -x \\ 2 & 1 \end{pmatrix}$ is

(A) $-5x - 3$

(B) $-x - 3$

(C) $1 - x^2$

(D) $4x$

(E) $2x + 2$

If $A = \begin{pmatrix} -3 & 1 & 6 \\ 2 & -5 & 0 \\ 1 & -3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 7 \\ -4 & 2 \\ -1 & -5 \end{pmatrix}$, and $AX = B$, then the size of X is

(A) 3 rows, 3 columns

(B) 3 rows, 2 columns

(C) 2 rows, 2 columns

(D) 2 rows, 3 columns
(E) cannot be determined

4. The chart below shows the number of small and large packages of a certain brand of cereal that were bought over a three-day period. The price of a small box of this brand is \$2.99, and the price of a large box is \$3.99. Which of the following matrix expressions represents the income, in dollars, received from the sale of cereal each of the three days?

	Day 1	Day 2	Day 3
Large	75	82	57
Small	43	36	50

(A) $\begin{pmatrix} 75 & 82 & 57 \\ 43 & 36 & 50 \end{pmatrix}(2.99 \quad 3.99)$

(B) $\begin{pmatrix} 75 & 43 \\ 82 & 36 \\ 57 & 50 \end{pmatrix}\begin{pmatrix} 3.99 \\ 2.99 \end{pmatrix}$

(C) $\begin{pmatrix} 75 & 82 & 57 \\ 43 & 36 & 50 \end{pmatrix}\begin{pmatrix} 2.99 \\ 3.99 \end{pmatrix}$

(D) $\begin{pmatrix} 2.99 \\ 3.99 \end{pmatrix}\begin{pmatrix} 75 & 43 \\ 82 & 36 \\ 57 & 50 \end{pmatrix}$

(E) $2.99\begin{pmatrix} 75 & 82 & 57 \\ 43 & 36 & 50 \end{pmatrix} + 3.99\begin{pmatrix} 75 & 82 & 57 \\ 43 & 36 & 50 \end{pmatrix}$

DETERMINANTS AND INVERSES OF SQUARE MATRICES

The determinant of an nby nsquare matrix is a number. The determinant of the 2 y 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is ad – bc.

:XAMPLE 1

Write an expression for the determinant of $\begin{pmatrix} 2 & -1 \\ 3 & x \end{pmatrix}$**.**

By definition, $\begin{vmatrix} 2 & -1 \\ 3 & x \end{vmatrix}$ = 2x – (–3) = 2x + 3.

XAMPLE 2

Solve for x: $\begin{vmatrix} x & x \\ 8 & x \end{vmatrix} = \begin{vmatrix} 7 & -2 & 1 \\ 0 & 3 & -1 \\ 5 & -4 & 2 \end{vmatrix}$

The determinant on the left side is x^2 – 8x Use the calculator to evaluate the terminant on the right as 9. This yields the quadratic equation x^2 – 8x – 9 = 0. can be solved by factoring to get x = 9 or x = –1.

larger square matrices, use the graphing calculator to calculate the minant (2nd MATRIX/MATH/det). A matrix whose determinant is zero is d singular. If the determinant is not zero, the matrix is nonsingular.
e product of square n by n matrices is a square n by n matrix. An identity is a square matrix consisting of 1's down the main diagonal and 0's e. The product of n by n square matrices I and A is A In other words, I ative identity for matrix multiplication.
lar square n by n matrix A has a multiplicative inverse, A^{-1}, where $A^{-1}A = AA^{-1} = I$ A
MATRIX/NAMES/A,
and ENTER.

EXAMPLE 3

If $A = \begin{pmatrix} 7 & -3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, **solve for X when AX = B**

latrix multiply both sides on the left by A^{-1}: $A^{-1}AX = A^{-1}B$ This yields $IX = X = A^{-1}B =$ $\begin{pmatrix} 26/31 \\ 9/31 \end{pmatrix}$. The fractional form of the answer can be obtained by keying MATH/ENTER/ENTER.

EXERCISES

1. The determinant of $\begin{pmatrix} p & 3 \\ -2 & 1 \end{pmatrix}$ is

(A) p– 6

(B) p+ 6

(C) 3p– 2

(D) 3 – 2p

(E) –6 – p

_ Find all values of xfor which $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \\ 4 & 1 & 6 \end{vmatrix} = \begin{vmatrix} x & 4 \\ 5 & x \end{vmatrix}$.

(A) ±3.78

(B) ±4.47

(C) ±5.12

(D) ±6.19

(E) ±6.97

3. If $X\begin{pmatrix} -7 & 2 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 5 & 4 \end{pmatrix}$, then X=

(A) $\begin{pmatrix} -\frac{2}{7} & -\frac{3}{2} \\ 0 & -\frac{5}{4} \end{pmatrix}$

(B) $\begin{pmatrix} -\frac{7}{2} & -\frac{2}{3} \\ 0 & -\frac{4}{5} \end{pmatrix}$

(C) $\begin{pmatrix} -\frac{2}{7} & \frac{17}{35} \\ -\frac{5}{7} & -\frac{38}{351} \end{pmatrix}$

(D) $\begin{pmatrix} -14 & -6 \\ 0 & -20 \end{pmatrix}$

(E) undefined

SOLVING SYSTEMS OF EQUATIONS

An important application of matrices is writing and solving systems of equations in matrix form.

EXAMPLE 1

Solve the system $\begin{array}{l} x - y + 2z = -3 \\ 2x + y - z = 0 \\ -x + 2y - 3z = 7 \end{array}$

This system can be written as AV = B, where A= $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & -3 \end{pmatrix}$ is the matrix of the coefficients, $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the matrix of the variables, and $B = \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}$ is the column matrix representing the right side of the system. Multiplying both sides of this equation on the left by A^{-1} yields $A^{-1}AV = A^{-1}B$, which reduces to V= A

given a system of equations, enter the matrix of coefficients into a matrix
ie column matrix of the right side into a second matrix (B), and find the $A^{-1}B$ to display the solution.

ERCISES

1. Find the matrix equation that represents the system $\begin{cases} 2x - 3 = 3y \\ y - 5x = 14 \end{cases}$

(A) $\begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$

(B) $\begin{pmatrix} 2 & -3 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$

(C) $\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$

(D) $\begin{pmatrix} 3 \\ 13 \end{pmatrix}\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}$

(E) This system cannot be represented as a matrix equation.

2. Find if $\begin{pmatrix} x \\ y \end{pmatrix}$ if $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$.

(A) (–2 0.5)

(B) $\begin{pmatrix} -5/6 \\ 1 \end{pmatrix}$

(C) (–1 3/4)

(D) $\begin{pmatrix} -2 \\ 1/2 \end{pmatrix}$

(E) (–5 –4/5)

Answers and Explanations

Addition, Subtraction, and Scalar Multiplication

1. **(C)** First find the sum of the matrices to the left of the equals sign: $\begin{pmatrix} 12 & 8 \\ -8 & 16 \end{pmatrix}$. Since the first row of the matrix to the right of the equals sign is (3 2), K must be 4. Since (J M) is the bottom row, $J = -2$ and $M =$

2. **(E)** In order for these matrices to be equal, $\begin{pmatrix} x & 2 \\ -3 & y \end{pmatrix} = \begin{pmatrix} 2x^2 & 2 \\ -3 & 6y-10 \end{pmatrix}$.

Therefore, $x = 2x^2$ and $y = 6y - 10$. Solving the first equation yields $x = 0$, $\frac{1}{2}$ and $y = 2$.

3. **(B)** To solve for X, first subtract $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 3 \end{pmatrix}$ from both sides of the equation.

Then $-X = \begin{pmatrix} 4 & 1 & -11 \\ -8 & 1 & -2 \end{pmatrix}$, so $X = \begin{pmatrix} -4 & -1 & 11 \\ 8 & -1 & 2 \end{pmatrix}$.

Matrix Multiplication

1. **(B)** By definition, $AB = \begin{pmatrix} (-2)(0)+(-1)(3)+(5)(4)+(9)(-6) \\ (-2)(-5)+(-1)(-2)+(5)(0)+(9)(1) \end{pmatrix} = \begin{pmatrix} -37 \\ 21 \end{pmatrix}$.
2. **(C)** By definition, the first row, second column of the product is $(x)(-x) + (1)(1) = -x^2 + 1$.
3. **(B)** X must have as many rows as A has columns, which is 3. X must have as many columns as B does, which is 2.
4. **(B)** Matrix multiplication is row by column. Since the answer must be a 3 by 1 matrix, the only possible answer choice is B.

Determinants and Inverses of Square Matrices

1. **(B)** By definition, the determinant of $\begin{pmatrix} p & 3 \\ -2 & 1 \end{pmatrix}$ is (p)(1) – (3)(–2) = p+ 6.
2. **(B)** Enter the 3 by 3 matrix on the left side of the equation into your graphing calculator and evaluate its determinant (zero). The determinant on the right side of the equation is $x^2 - 20$. Therefore x= $\pm\ \sqrt{20} \approx \pm 4.47$.

3. **(C)** To find X, multiply both sides of the equation by $\begin{pmatrix} -7 & 2 \\ 0 & -5 \end{pmatrix}^{-1}$ on the right. Enter both matrices in your calculator, key the product $\begin{pmatrix} 2 & -3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} -7 & 2 \\ 0 & -5 \end{pmatrix}^{-1}$ on your graphing calculator, and key MATH/ENTER/ENTER to convert the decimal answer to a fraction.

Solving Systems of Equations

1. **(B)** First, write the system in standard form: $\begin{cases} 2x - 3y = 3 \\ -5x + y = 14 \end{cases}$. The matrix form of this equation is $\begin{pmatrix} 2 & -3 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$.

2. **(D)** This is the matrix form AX = B of a system of equations. Multiply both sides of the equation by A^{1} on the left to get the solution X= $\begin{pmatrix} x \\ y \end{pmatrix} = A^{1}B$. Enter the 2 by 2 matrix, A, and the 2 by 1 matrix, B, into your graphing calculator. Return to the home screen and enter $A^{1}B=$ $\begin{pmatrix} -2 \\ 0.5 \end{pmatrix}$.

Sequences and Series

ARITHMETIC SEQUENCES

One of the most common sequences studied at this level is an arithmetic equence (or arithmetic progression). Each term differs from the preceding rm by a common difference. The first n terms of an arithmetic sequence can be noted by $t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots, t_1 + (n - 1)d$

on difference and $t_n = t_1 + (n - 1)d$ The sum of n terms of n an arithmetic sequence is given by the formula

$$S_n = \frac{n}{2}(t_1 + t_n) \quad \text{or} \quad S_n = \frac{n}{2}[2t_1 + (n-1)d]$$

If there is one term falling between two given terms of an arithmetic sequence, it is called their arithmetic mean.

EXAMPLE 1

(A) Find the 28th term of the arithmetic sequence 2, 5, 8, ...

(B) Express the sum of 28 terms of the series of this sequence using sigma notation.

(C) Find the sum of the first 28 terms of the series.

SOLUTIONS

(A) $t_n = t_1 + (n-1)d$

$t_1 = 2$, $d = 3$, $n = 28$ $t_{28} = 2 + 27 \cdot 3 = 83$

(B) $\sum_{k=0}^{27}(3k+2)$ or $\sum_{j=1}^{28}(3j-1)$

(C) $S_n = \frac{n}{2}(t_1 + t_n)$

$\frac{28}{2}(2 + 83) = 14 \cdot 85 = 1190$

EXAMPLE 2

If $t_8 = 4$ and $t_{12} = -2$, find the first three terms of the arithmetic sequence.

$$t_n = t_1 + (n-1)d$$
$$4 = t_1 + 7d$$
$$-2 = t_1 + 11d$$

To solve these two equations for d, subtract the second equation from the third.

$$-6 = 4d$$

$$d = -\frac{3}{2}$$

Substituting in the first equation gives $4 = t_1 + 7\left(-\frac{3}{2}\right)$. Thus,

$$t_1 = 4 + \frac{21}{2} = \frac{29}{2}$$
$$t_2 = \frac{29}{2} + \left(-\frac{3}{2}\right) = \frac{26}{2} = 13$$
$$t_3 = \frac{29}{2} + 2\left(-\frac{3}{2}\right) = \frac{23}{2}$$

The first three terms are $\frac{29}{2}, 13, \frac{23}{2}$.

EXAMPLE 3

In an arithmetic series, if $S_n = 3n^2 + 2n$, find the first three terms.

n n= 1, $S_1 = t_1$. Therefore, $t_1 = 3(1)^2 + 2 \cdot 1 = 5$.

$$S_2 = t_1 + t_2 = 3(2)^2 + 2 \cdot 2 = 16$$
$$5 + t_2 = 16$$
$$t_2 = 11$$

Therefore, $d = 6$, which leads to a third term of 17. Thus, the first three terms are 5, 11, 17.

GEOMETRIC SEQUENCES

Another common type of sequence studied at this level is a geometric sequence (or geometric progression). In a geometric sequence the ratio of any two successive terms is a constant r called the constant ratio. The first n terms of a geometric sequence can be denoted by $t_1, t_1r, t_1r^2, t_1r^3, \ldots, t_1r^{n-1} = t_n$

The sum of the first n terms of a geometric series is given by the formula

$$S_n = \frac{t_1\left(1-r^n\right)}{1-r}$$

If there is one term falling between two given terms of a geometric sequence it is called their geometric mean.

EXAMPLE 1

(A) Find the seventh term of the geometric sequence 1, 2, 4, . . . , and (B) the sum of the first seven terms.

(A) $r = \frac{t_2}{t_1} = \frac{2}{1} = 2; t_7 = t_1 r^{7-1}; t_7 = 1 \cdot 2^6 = 64$

(B) $S_7 = \frac{1(1-2^7)}{1-2} = \frac{1-128}{-1} = 127$

EXAMPLE 2

The first term of a geometric sequence is 64, and the common ratio is $\frac{1}{4}$. For what value of nis $t_n = \frac{1}{4}$?

$$\frac{1}{4} = 64\left(\frac{1}{4}\right)^{n-1}$$

$$\left(\frac{1}{4}\right)^{2-n} = 64 = 4^3$$

$$4^{n-2} = 4^3$$

$$n = 5$$

SERIES

In a geometric sequence, if |r| < 1, the sum of the series approaches a limit as n pproaches infinity. In the formula $S_n = \frac{t_1(1-r^n)}{1-r}$, if $|r| < 1$, the term $r^n \to 0$ as $n \to \infty$. Therefore, as long as $|r| < 1$, $\lim_{n\to\infty} S_n = \frac{t_1}{1-r}$, or $S = \frac{t_1}{1-r}$

EXAMPLE 1

Evaluate (A) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k}$ and

(B) $\sum_{j=0}^{\infty} (-3)^{-j}$

Both problems ask the same question: Find the sum of an infinite geometric series.

(A) When the first few terms, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, are listed, it can be seen that $t_1=\frac{1}{2}$ and the common ratio $r=\frac{1}{2}$. Therefore,

$$S=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1.$$

(B) When the first few terms, $\frac{1}{1}-\frac{1}{3}+\frac{1}{9}-\cdots$, are listed, it can be seen that $t_1 = 1$

nd the common ratio $r=-\frac{1}{3}$. Therefore,

$$S=\frac{1}{1-\left(-\frac{1}{3}\right)}=\frac{1}{\frac{4}{3}}=\frac{3}{4}.$$

EXAMPLE 2

Find the exact value of the repeating decimal 0.4545 . . . .

This can be represented by a geometric series, 0.45 + 0.0045 + 0.000045 + · · · , with t_1 = 0.45 and r= 0.01.

Since $|r|<1$,

$$S=\frac{0.45}{1-0.01}=\frac{0.45}{0.99}=\frac{45}{99}=\frac{5}{11}.$$

EXAMPLE 3

Given the sequence 2, x, y, 9. If the first three terms form an arithmetic sequence and the last three terms form a geometric sequence, find xand y.

m the arithmetic sequence, $\begin{cases} x=2+d \\ y=2+2d \end{cases}$, substitute to eliminate d

$$y=2+2(x-2)$$
$$y=2+2x-4$$
$$*y=2x-2$$

From the geometric sequence $\begin{cases} 9=yr \\ y=xr \end{cases}$, substitute to eliminate r

$$9 = y \cdot \frac{y}{x}$$

$$*\,9x = y^2$$

Use the two equations with the * to eliminate y:

$$9x = (2x-2)^2$$

$$9x = 4x^2 - 8x + 4$$

$$4x^2 - 17x + 4 = 0$$

$$(4x-1)(x-4) = 0$$

$$4x - 1 = 0 \quad \text{or} \quad x - 4 = 0$$

Thus, $x = \frac{1}{4}$ or 4.

Substitute in $y = 2x - 2$:

$$\text{if } x = \frac{1}{4},\ y = -\frac{3}{2}$$

$$\text{if } x = 4,\ y = 6.$$

EXERCISES

1. If $a_1 = 3$ and $a_n = n + a_{n-1}$, the sum of the first five terms is
(A) 17
(B) 30
(C) 42
(D) 45
(E) 68

_ If $a_1 = 5$ and $a_n = 1 + \sqrt{a_{n-1}}$, find a_3.
(A) 2.623
(B) 2.635
(C) 2.673
(D) 2.799
(E) 3.323

3. If the repeating decimal $0.23\overline{737}\ldots$ is written as a fraction in lowest terms, the sum of the numerator and denominator is
(A) 16
(B) 47
(C) 245
(D) 334
(E) 1237

4. The first three terms of a geometric sequence are $\sqrt[4]{3}, \sqrt[8]{3}, 1$. The fourth term is
(A) $\sqrt[32]{3}$
(B) $\sqrt[16]{3}$

(C) $\frac{1}{\sqrt[16]{3}}$

(D) $\frac{1}{\sqrt[8]{3}}$

(E) $\frac{1}{\sqrt[4]{3}}$

5. By how much does the arithmetic mean between 1 and 25 exceed the positive geometric mean between 1 and 25?

(A) 5
(B) about 7.1
(C) 8
(D) 12.9
(E) 18

6. In a geometric series $S = \frac{2}{3}$ and $t_1 = \frac{2}{7}$. What is r?

(A) $\frac{2}{3}$

(B) $-\frac{4}{7}$

(C) $\frac{2}{7}$

(D) $\frac{4}{7}$

(E) $-\frac{2}{7}$

Answers and Explanations

1. **(D)** $a_2 = 5$, $a_3 = 8$, $a_4 = 12$, $a_5 = 17$. Therefore, $S_5 = 45$

2. **(D)** Press 5 ENTER into your graphing calculator. Then enter $1+\sqrt{\text{Ans}}$ and press ENTER twice more to get a_3.

3. **(C)** The decimal $0.23\overline{737}$ = 0.2 + (0.037 + 0.00037 + 0.0000037 + · · ·), which is 0.2 + an infinite geometric series with a common ratio of 0.01.

$$S_n = 0.2 + \frac{0.037}{0.99} = \frac{2}{10} + \frac{37}{990} = \frac{235}{990} = \frac{47}{198}.$$

The sum of the numerator and the denominator is 245.

4. **(D)** Terms are $3^{1/4}$, $3^{1/8}$, 1. Common ratio = $3^{-1/8}$. Therefore, the fourth term is $1 \cdot 3^{-1/8} = 3^{-1/8}$ or $\frac{1}{\sqrt[8]{3}}$.

5. **(C)** Arithmetic mean $= \frac{1+25}{2} = 13.$ Geometric mean $= \sqrt{1 \cdot 25} = 5.$ The difference is 8.

6. **(D)** $\frac{2}{3}=\frac{\frac{2}{7}}{1-r}$. $2-2r=\frac{6}{7}$. 14 – 14r = 6. Therefore, $r=\frac{4}{7}$.

Functions

Functions

Overview

DEFINITIONS

A *relation* is a set of ordered pairs. A *function* is a relation such that for each first element there is one and only one second element. The set of numbers that make up all first elements of the ordered pairs is called the *domain* of the function, and the resulting set of second elements is called the *range* of the function.

EXAMPLE 1

{(1,2),(3,4),(5,6),(6,1),(2,2)}

This is a function because every ordered pair has a different first element. Domain = {1,2,3,5,6}. Range = {1,2,4,6}.

EXAMPLE 2

$f(x) = 3x + 2$

This is a function because for each value substituted for *x* there is one and only one value for $f(x)$. Domain = {all real numbers}. Range = {all real numbers}.

EXAMPLE 3

{(1,2),(3,2),(1,4)}

This is a relation but *not* a function because when the first element is 1, the second element can be either 2 or 4. Domain = {1,3}. Range = {2,4}.

EXAMPLE 4

$\{(x,y):y^2 = x\}$

[This should be read, "The set of all ordered pairs (x,y) such that $y^2 = x$."] This is a relation but *not* a function because, for each nonnegative number that is substituted for x, there are two values for y. For example, $x = 4$, $y = +2$, or $y = -2$. Domain = {all nonnegative real numbers}. Range = {all real numbers}.

The expressions $f = \{(x,y):y = x^2\}$ and $f(x) = x^2$ both name the same function; f is the rule that pairs any number with its square. Thus, $f(x) = x^2$, $f(a) = a^2$, $f(z) = z^2$ all name the same function. The symbol $f(2)$ is the value of the function f when $x = 2$. Thus, $f(2) = 4$.

EXERCISES

1. If {(3,2),(4,2),(3,1),(7,1),(2,3)} is to be a function, which one of the following must be removed from the set?
   (A) (3,2)
   (B) (4,2)
   (C) (2,3)
   (D) (7,1)
   (E) none of the above
2. For $f(x) = 3x^2 + 4$, $g(x) = 2$, and h {(1,1), (2,1), (3,2)},
   (A) f is the only function
   (B) h is the only function
   (C) f and g are the only functions
   (D) g and h are the only functions
   (E) f, g, and h are all functions

3. What value(s) must be excluded from the domain of $f = \left\{(x, y): y = \dfrac{x+2}{x-2}\right\}$?

(A) –2
(B) 0
(C) 2
(D) 2 and –2
(E) no value

TIP

Looking for answers? All answers to exercises appear at the end of each section. Resist the urge to peek before trying the problems on your own.

COMBINING FUNCTIONS

Given two functions, *f* and *g*, five new functions can be defined:

Sum function	$(f+g)(x) = f(x) + g(x)$
Difference function	$(f-g)(x) = f(x) - g(x)$
Product function	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient function	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

if and only if $g(x) \neq 0$

Composition of functions $(f \circ g)(x) = f(g(x))$

EXAMPLE

If $f(x) = 3x - 2$ and $g(x) = x^2 - 4$, then indicate the function that represents

(A) $(f+g)(x)$
(B) $(f-g)(x)$
(C) $f \cdot g\,(x)$
(D) $\dfrac{f}{g}(x)$
(E) $(f \circ g)(x)$
(F) $(g \circ f)(x)$

SOLUTIONS

(A)
$$(f+g)(x) = f(x) + g(x)$$
$$= (3x-2) + (x^2-4) = x^2 + 3x - 6$$

(B)
$$(f-g)(x) = f(x) - g(x)$$
$$= (3x-2) - (x^2-4) = -x^2 + 3x + 2$$

(C)
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$= (3x-2)(x^2-4)$$
$$= 3x^3 - 2x^2 - 12x + 8$$

(D)
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3x-2}{x^2-4} \text{ and } x \neq \pm 2$$

(E)
$$(f \circ g)(x) = f(x) \circ g(x)$$
$$= f(g(x)) = 3(g(x)) - 2$$
$$= 3(x^2-4) - 2 = 3x^2 - 14$$

(F)
$$(g \circ f)(x) = g(x) \circ f(x)$$
$$= g(f(x)) = (f(x))^2 - 4$$
$$= (3x-2)^2 - 4 = 9x^2 - 12x$$

TIP

$(f \circ g)(x)$ and $(g \circ f)(x)$ *need not be* the same!

EXERCISES

1. If $f(x) = 3x^2 - 2x + 4$, $f(-2) =$
   - **(A)** –12
   - **(B)** –4
   - **(C)** –2
   - **(D)** 12
   - **(E)** 20
2. If $f(x) = 4x - 5$ and $g(x) = 3^x$, then $f(g(2)) =$
   - **(A)** 3
   - **(B)** 9
   - **(C)** 27
   - **(D)** 31

(E) none of the above

3. If $f(g(x)) = 4x^2 - 8x$ and $f(x) = x^2 - 4$, then $g(x) =$

(A) $4 - x$
(B) x
(C) $2x - 2$
(D) $4x$
(E) x^2

4. What values must be excluded from the domain of $\left(\frac{f}{g}\right)(x)$ if $f(x) = 3x^2 - 4x + 1$ and $g(x) = 3x^2 - 3$?

(A) 0
(B) 1
(C) 3
(D) both ± 1
(E) no values

5. If $g(x) = 3x + 2$ and $g(f(x)) = x$, then $f(2) =$

(A) 0
(B) 1
(C) 2
(D) 6
(E) 8

6. If $p(x) = 4x - 6$ and $p(a) = 0$, then $a =$

(A) -6
(B) $-\frac{3}{2}$
(C) $\frac{3}{2}$
(D) $\frac{2}{3}$
(E) 2

7. If $f(x) = e^x$ and $g(x) = \sin x$, then the value of $(f \circ g)(\sqrt{2})$ is

(A) –0.01
(B) –0.8
(C) 0.34
(D) 1.8
(E) 2.7

INVERSES

The *inverse* of a function f, denoted by f^{-1}, is a relation that has the property that $f(x) \circ f^{-1}(x) = f^{-1}(x) \circ f(x) = x$, where f^{-1} is not necessarily a function.

EXAMPLE 1

$f(x) = 3x + 2$. Is $y = \dfrac{x-2}{3}$ the inverse of f?

To answer this question assume that $f^{-1}(x) = \dfrac{x-2}{3}$ and verify that $f(x) \circ f^{-1}(x) = x$.
To verify this, proceed as follows:

$$f(x)\circ f^{-1}(x) = f(f^{-1}(x))$$
$$= f\left(\frac{x-2}{3}\right) = 3\left(\frac{x-2}{3}\right) + 2 = x$$

and

$$f^{-1}(x)\circ f(x) = f^{-1}(f(x)) = f^{-1}(3x+2)$$
$$= \frac{(3x+2)-2}{3} = x.$$

Since $f(x) \circ f^{-1}(x) \circ f^{-1}(x) \circ f(x) = x$, $\dfrac{x-2}{3}$ is the inverse of f.

EXAMPLE 2

f = {(1,2),(2,3),(3,2)}. Find the inverse.

f^{-1} ={(2,1), (3,2), (2,3)}

> **TIP**
>
> Remember that the inverse of a function need not be a function.

To verify this, check $f \circ f^{-1}$ and $f^{-1} \circ f$ term by term.

$(f\circ f^{-1})^{(x)} = f(f^{-1}(x))$; when $x = 2, f(f^{-1}(2)) = f(1) = 2$

when $x = 3, f(f^{-1}(3)) = f(2) = 3$

when $x = 2, f(f^{-1}(2)) = f(3) = 2$

Thus, for each x, $f(f^{-1}(x)) = x$.

$(f^{-1}\circ f)^{(x)} = f^{-1}\circ(f(x))$; when $x = 1, f^{-1}(f(1)) = f^{-1}(2) = 1$

when $x = 2, f^{-1}(f(2)) = f^{-1}(3) = 2$

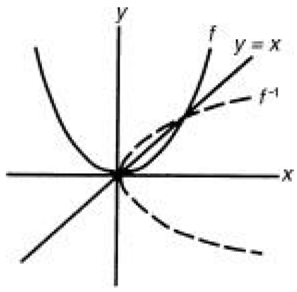
when $x = 3, f^{-1}(f(3)) = f^{-1}(2) = 3$

Thus, for each x, $f^{-1}(f(x)) = x$. In this case f^{-1} is *not* a function.
If the point with coordinates (a,b) belongs to a function f, then the point with

coordinates (b,a) belongs to the inverse of f. Because this is true of a function and its inverse, the graph of the inverse is the reflection of the graph of f about the line $y = x$.

EXAMPLE 3

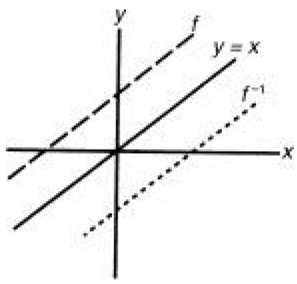
f^{-1} is *not* a function.



> **TIP**
>
> Graphs of inverses are reflections about the line $y = x$.

EXAMPLE 4

f^{-1} is a function.



As can be seen from the above examples, if the graph of f is given, the graph of f^{-1} is the image obtained by folding the graph of f about the line $y = x$. Algebraically the equation of the inverse of f can be found by interchanging the variables.

EXAMPLE 5

$f = \{(x,y): y = 3x + 2\}$. Find f^{-1}.

In order to find f^{-1}, interchange x and y and solve for y: $x = 3y + 2$, which becomes $y = \frac{x-2}{3}$.

Thus,

$$f^{-1} = \left\{(x,y) : y = \frac{x-2}{3}\right\}$$

EXAMPLE 6

$f = \{(x,y) : y = x^2\}$. Find f^{-1}.

Interchange x and y: $x = y^2$.

Solve for $y = y = \pm\sqrt{x}$.

Thus, $f^{-1} = \left\{(x,y) : y = \pm\sqrt{x}\right\}$, which is *not* a function.

The inverse of any function f can always be made a function by limiting the domain of f. In Example 6 the domain of f could be limited to all nonnegative numbers or all nonpositive numbers. In this way f^{-1} would become either $y = +\sqrt{x}$ or $y = -\sqrt{x}$, both of which are functions.

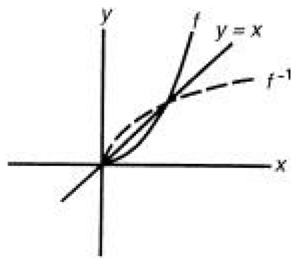
EXAMPLE 7

$f = \{(x,y) : y = x^2 \text{ and } x \geq 0\}$. Find f^{-1}.

$f^{-1} = \{(x,y) : x = y^2 \text{ and } y \geq 0\}$, which can also be written as

$$f^{-1} = \left\{(x,y) : y = +\sqrt{x}\right\}.$$

Here f^{-1} is a function.

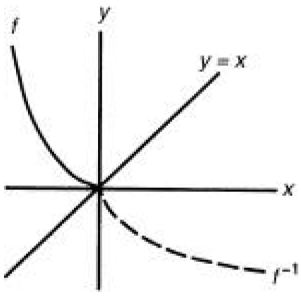


EXAMPLE 8

$f = \{(x,y): y = x^2 \text{ and } x \le 0\}$. Find f^{-1}.

$f^{-1} = \{(x,y): x = y^2 \text{ and } y \le 0\}$, which can also be written as
$f^{-1} = \{(x,y): y = -\sqrt{x}\}$.

Here f^{-1} is a function.



EXERCISES

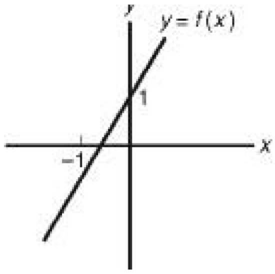
1. If $f(x) = 2x - 3$, the inverse of f, f^{-1}, could be represented by
   (A) $f^{-1}(x) = 3x-2$
   (B) $f^{-1}(x) = \dfrac{1}{2x-3}$
   (C) $f^{-1}(x) = \dfrac{x-2}{3}$
   (D) $f^{-1}(x) = \dfrac{x+2}{3}$
   (E) $f^{-1}(x) = \dfrac{x+3}{2}$

2. If $f(x) = x$, the inverse of f, f^{-1}, could be represented by
   (A) $f^{-1}(x) = x$
   (B) $f^{-1}(x) = 1$
   (C) $f^{-1}(x) = \dfrac{1}{x}$
   (D) $f^{-1}(x) = y$
   (E) f^{-1} does not exist

3. The inverse of $f = \{(1,2),(2,3),(3,4),(4,1),(5,2)\}$ would be a function if the domain of f is limited to
   (A) $\{1,3,5\}$
   (B) $\{1,2,3,4\}$
   (C) $\{1,5\}$
   (D) $\{1,2,4,5\}$
   (E) $\{1,2,3,4,5\}$

4. Which of the following could represent the equation of the inverse of the

graph in the figure?



(A) $y = -2x + 1$
(B) $y = 2x + 1$
(C) $y = \frac{1}{2}x + 1$
(D) $y = \frac{1}{2}x - 1$
(E) $y = \frac{1}{2}x - \frac{1}{2}$

ODD AND EVEN FUNCTIONS

A relation is said to be *even* if $(-x,y)$ is in the relation whenever (x,y) is. If the relation is defined by an equation, it is even if $(-x,y)$ satisfies the equation whenever (x,y) does. If the relation is a function f, it is even if $f(-x) = f(x)$ for all x in the domain of f. The graph of an even relation or function is symmetric with respect to the y axis.

EXAMPLE 1

$\{(1,0),(-1,0),(3,0),(-3,0),(5,4),(-5,4)\}$ is an even relation because $(-x,y)$ is in the relation whenever (x,y) is.

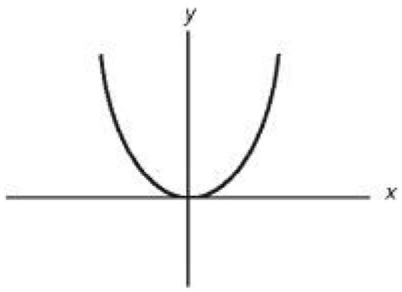
EXAMPLE 2

$x^4 + y^2 = 10$ is an even relation because $(-x)^4 + y^2 = x^4 + y^2 = 10$.

> **TIP**
>
> An even relation is symmetric about the y-axis.

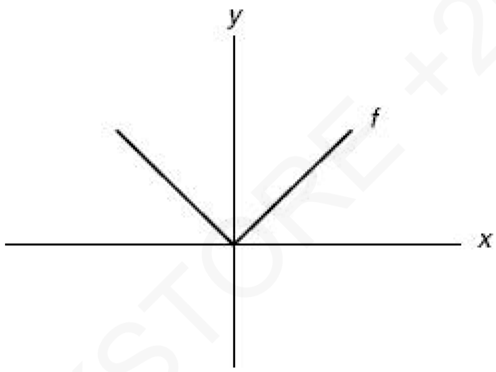
EXAMPLE 3

$$f(x) = x^2 \text{ and } f(-x) = (-x)^2 = x^2.$$



EXAMPLE 4

$$f(x) = |\,x\,| \text{ and } f(-x) = |\,-x\,| = |\,-1 \cdot x\,| = |\,-1\,| \cdot |\,x\,| = |\,x\,|.$$



A relation is said to be *odd* if $(-x,-y)$ is in the relation whenever (x,y) is. If the relation is defined by an equation, it is odd if $(-x,-y)$ satisfies the equation whenever (x,y) does. If the relation is a function f, it is odd if $f(-x) = -f(x)$ for all x in the domain of x. The graph of an odd relation or function is symmetric with respect to the origin.

EXAMPLE 5

$\{(5,3),(-5,-3),(2,1),(-2,-1),(-10,8),\ (10,-8)\}$ is an odd relation because $(-x,-y)$ is in the relation whenever (x,y) is.

EXAMPLE 6

$x^4 + y^2 = 10$ is an odd relation because $(-x)^4 + (-y)^2 = x^4 + y^2 = 10$. Note that $x^4 + y^2 = 10$ is both even and odd.

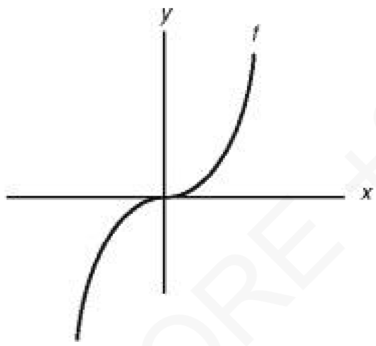
TIP

An odd relation is symmetric about the origin.

EXAMPLE 7

$$f(x) = x^3 \text{ and } f(-x) = (-x)^3 = -x^3.$$

Therefore, $f(-x) = x^3 = -f(x)$.



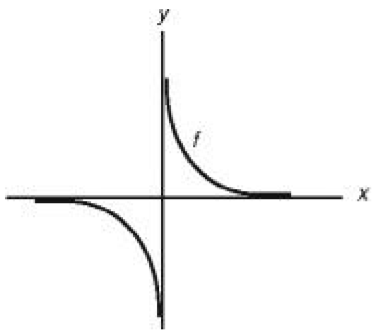
TIP

Relations can be either odd, even, or neither. They can also be both odd and even!

EXAMPLE 8

$f(x) = \frac{1}{x}$ and $f(-x) = \frac{1}{-x}$.

Therefore, $f(-x) = \frac{1}{x} = -f(x)$.



The sum of even functions is even. The sum of odd functions is odd. The product of an even function and an odd function is odd.

EXERCISES

1. Which of the following relations are *even*?

I. $y = 2$
II. $f(x) = x$
III. $x^2 + y^2 = 1$

(A) only I
(B) only I and II
(C) only II and III
(D) only I and III
(E) I, II, and III

2. Which of the following relations are *odd*?

I. $y = 2$
II. $y = x$
III. $x^2 + y^2 = 1$

(A) only II
(B) only I and II
(C) only I and III
(D) only II and III
(E) I, II, and III

3. Which of the following relations are both *odd* and *even*?

I. $x^2 + y^2 = 1$
II. $x^2 - y^2 = 0$
III. $x + y = 0$

(A) only III
(B) only I and II
(C) only I and III
(D) only II and III
(E) I, II, and III

4. Which of the following functions is neither *odd* nor *even*?

(A) $\{(1,2),(4,7),(-1,2),(0,4),(-4,7)\}$
(B) $\{(1,2),(4,7),(-1,-2),(0,0),(-4,-7)\}$
(C) $\{(x,y): y = x^3 - 1\}$
(D) $\{(x,y): y = x^2 - 1\}$

(E) $f(x) = -x$

Answers and Explanations

Definitions

1. **(A)** Either (3,2) or (3,1), which is not an answer choice, must be removed so that 3 will be paired with only one number.
2. **(E)** For each value of x there is only one value for y in each case. Therefore, f, g, and h are all functions.
3. **(C)** Since division by zero is forbidden, x cannot equal 2.

Combining Functions

1. **(E)** $f(-2) = 3(-2)^2 - 2(-2) + 4 = 20$.
2. **(D)** $g(2) = 3^2 = 9$. $f(g(2)) = f(9) = 31$.
3. **(C)** To get from $f(x)$ to $f(g(x))$, x^2 must become $4x^2$. Therefore, the answer must contain $2x$ since $(2x)^2 = 4x^2$.
4. **(D)** $g(x)$ cannot equal 0. Therefore, $x \neq \pm 1$.
5. **(A)** Since $f(2)$ implies that $x = 2$, $g(f(2)) = 2$. Therefore, $g(f(2)) = 3(f(2)) + 2 = 2$. Therefore, $f(2) = 0$.
6. **(C)** $p(a) = 0$ implies $4a - 6 = 0$, so $a = \frac{3}{2}$.
7. **(E)** $f \circ g(\sqrt{2}) = f(g(\sqrt{2})) = f(\sin\sqrt{2}) = e^{\sin\sqrt{2}} \approx 2.7$.

Inverses

1. **(E)** If $y = 2x - 3$, the inverse is $x = 2y - 3$, which is equivalent to $y = \frac{x+3}{2}$.
2. **(A)** By definition.
3. **(B)** The inverse is {(2,1),(3,2),(4,3),(1,4),(2,5)}, which is not a function because of (2,1) and (2,5). Therefore, the domain of the original function must lose either 1 or 5.
4. **(E)** If this line were reflected about the line $y = x$ to get its inverse, the slope would be less than 1 and the y-intercept would be less than zero. The only possibilities are Choices D and E. Choice D can be excluded because since the x-intercept of $f(x)$ is greater than –1, the y-intercept of its inverse must be greater than –1.

Odd and Even Functions

1. **(D)** Use the appropriate test for determining whether a relation is even.

I. The graph of $y = 2$ is a horizontal line, which is symmetric about the y-axis, so $y = 2$ is even.

II. Since $f(-x) = -x \neq x = f(x)$ unless $x = 0$, this function is not even.

III. Since $(-x)^2 + y^2 = 1$ whenever $x^2 + y^2 = 1$, this relation is even.

2. **(D)** Use the appropriate test for determining whether a relation is odd.

I. The graph of $y = 2$ is a horizontal line, which is not symmetric about the origin, so $y = 2$ is not odd.

II. Since $f(-x) = -x = -f(x)$, this function is odd.

III. Since $(-x)^2 + (-y)^2 = 1$ whenever $x^2 + y^2 = 1$, this relation is odd.

3. **(B)** The analysis of relation III in the above examples indicates that I and II are both even and odd. Since $-x + y \neq 0$ when $x + y = 0$ unless $x = 0$, III is not even, and is therefore not both even and odd.

4. **(C)** A is even, B is odd, D is even, and E is odd. C is not even because $(-x)^3 - 1 = -x^3 - 1$, which is neither $x^3 - 1$ nor $-x^3 + 1$.

INEQUALITIES

Given any algebraic expression $f(x)$, there are exactly three situations that can exist:

1. for some values of x, $f(x) < 0$;
2. for some values of x, $f(x) = 0$;
3. for some values of x, $f(x) > 0$.

If all three of these sets of numbers are indicated on a number line, the set of values that satisfy $f(x) < 0$ is always separated from the set of values that satisfy $f(x) > 0$ by the values of x that satisfy $f(x) = 0$.

EXAMPLE

Find the set of values for x that satisfies $x^2 - 3x - 4 < 0$.

Graph $y = x^2 - 3x - 4$. You need to find the x values of points on the graph that lie below the x-axis. First find the zeros: $x = 4$, $x = -1$. The points that lie below the x-axis are (strictly) between -1 and 4, or $-1 < x < 4$.

EXERCISES

1. Which of the following is equivalent to $3x^2 - x < 2$?

(A) $-\frac{3}{2} < x < 1$

(B) $-1 < x < \frac{2}{3}$

(C) $-\frac{2}{3} < x < 1$

(D) $-1 < x < \frac{3}{2}$

(E) $x < -\frac{2}{3}$ or $x > 1$

2. Solve $x^5 - 3x^3 + 2x^2 - 3 > 0$.

(A) $(-\infty, -0.87)$

(B) $(-1.90, -0.87)$

(C) $(-1.90, -0.87) \cup (1.58, \infty)$

(D) $(-0.87, 1.58)$

(E) $(1.58, \infty)$

3. The number of integers that satisfy the inequality $x^2 + 48 < 16x$ is

(A) 0

(B) 4

(C) 7

(D) an infinite number

(E) none of the above

Answers and Explanations

Inequalities

1. (C) $3x^2 - x - 2 = (3x + 2)(x - 1) = 0$ when $x = -\frac{2}{3}$ or 1. Numbers between these satisfy the original inequality.

2. **(C)** Graph the function, and determine that the three zeros are –1.90, –0.87, and 1.58. The parts of the graph that are above the x-axis have x-coordinates between –1.90 and –0.87 and are larger than 1.58.

3. **(C)** $x^2 - 16x + 48 = (x - 4)(x - 12) = 0$, when $x = 4$ or 12. Numbers between these satisfy the original inequality.

PLANE GEOMETRY

Lines and Angles

On the Math 1 test, approximately 20 of the 50 questions deal with geometry. Therefore, mastering geometry is crucial for anyone who is taking this test. Of the 20 geometry questions, about 10 of them are on plane geometry; the other 10 are split between solid geometry and coordinate geometry, which will be discussed in Chapters
Much of the geometry you have learned in your math classes is not covered on the Math 1 test. In this chapter, we will review all of the important topics in plane geometry that you need to know for the Math 1 test and only those.

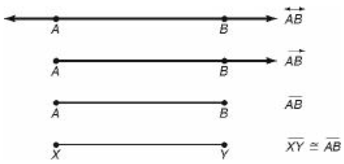
Lines are usually referred to by lowercase letters, such as ℓ, m, and n. We can also name a line using two of the points on the line. If A and B are points on line ℓ, we can refer to line ℓ as line $\overline{AB}$.

$\overline{AB}$ represents the ***ray*** that consists of point A and all the points on $\overline{AB}$ that are on the same side of A as B.

$\overline{AB}$ represents the ***line segment*** that consists of points A and B and all the points on $\overline{AB}$ that are between them.

Finally, AB represents the ***length*** of segment $\overline{AB}$.

If two line segments have the same length, we say they are ***congruent***. The symbol $\cong$ is used to indicate congruence, so in the figure below we have $\overline{AB} \cong \overline{XY}$.



TIP

$\overline{AB} \cong \overline{XY}$ the same thing as $AB = XY$.

ANGLES

An ***angle*** is formed by the intersection of two line segments, rays, or lines. The point of intersection is called the ***vertex*** of the angle. On the Math 1 test, angles are always measured in degrees and are classified according to their degree measures.

Key Fact G1

- **An *acute* angle measures less than 90º**

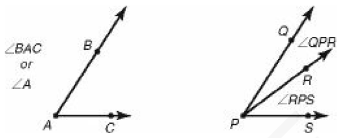
- **A *right* angle measures 90º**

- **An *obtuse* angle measures more than 90º but less than 180º.**

- **A *straight* angle measures 180º.**

> **Note**
>
> A small square drawn at the vertex of an angle is used to indicate that the angle is a right angle. On the Math Level 1 test, if an angle has a square in it, it must be a 90° angle, *even if the figure has not been drawn to scale.*

An angle can be named by three points: a point on one side, the vertex, and a point on the other side, in that order. When there is no possible ambiguity, we can name the angle just by its vertex.

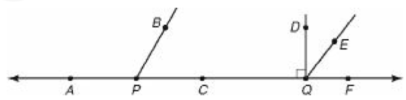


To indicate that the measure of angle A is 60°, say, we write $m\angle A = 60°$.

Two angles that have the same measure are said to be ***congruent***, denoted by the symbol ≅. So if $m\angle A = m\angle B$, we can write $\angle A \cong \angle B$.

> **TIP**
>
> $\angle A \cong \angle B$ means the same thing as $m\angle A = m\angle B$

If the sum of the measures of two angles is 180°, we say that the angles are ***supplementary***. If the sum of their measures is 90°, the angles are ***complementary***. In the diagram below, $\angle APB$ is supplementary to $\angle BPC$ and $\angle DQE$ is complementary to $\angle EQF$.



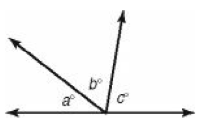
Key Fact G2

If two or more angles form a straight angle, the sum of their measures is 180º.

EXAMPLE 1: If in the figure below $a : b : c = 2 : 3 : 4$, then $a = 2x$, $b = 3x$, and $c = 4x$. So

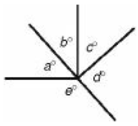
$$2x + 3x + 4x = 180 \Rightarrow 9x = 180 \Rightarrow$$

$$x = 20° \Rightarrow a = 40°,\ b = 60°, \text{ and } c = 80°$$



Key Fact G3

The sum of the measures of all nonoverlapping angles around a point is 360°.



$$a + b + c + d + e = 360$$

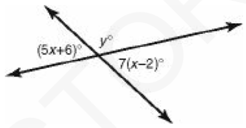
When two lines intersect, four angles are formed. The two angles in each pair of opposite angles are called ***vertical angles***.

Key Fact G4

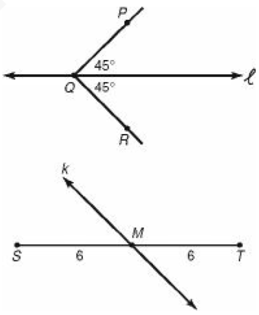
Vertical angles have equal measures.

> **TIP**
>
> Key Fact G4 means that vertical angles are congruent.

EXAMPLE 2: To find the value of y in the figure below, note that $5x + 6 = 7(x - 2) \Rightarrow 5x + 6 = 7x - 14 \Rightarrow 2x = 20$. Therefore, $x = 10$. So the measure of each acute angle is $(5 \times 10 + 6)° = 56°$ and $y = 180 - 56 = 124$.



In the figure below, line ℓ divides $\angle PQR$ into two equal parts, and line k divides line segment $\overline{ST}$ into two equal parts. Line ℓ is said to ***bisect*** the angle, and line k ***bisects*** the line segment. Point M is called the ***midpoint*** of segment $\overline{ST}$.

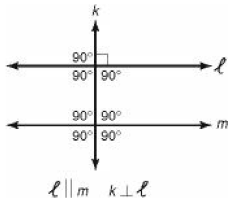


PERPENDICULAR AND PARALLEL LINES

Two lines that intersect to form right angles are called ***perpendicular***. Two lines that never intersect are said to be ***parallel***. Consequently, parallel lines form no angles. However, if a third line, called a ***transversal***, intersects a pair of parallel lines, eight angles are formed, and the relationships between these angles are very important.

Key Fact G5

If a pair of parallel lines is cut by a transversal that is perpendicular to the parallel lines, all eight angles are right angles.



Key Fact G6

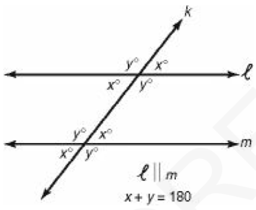
If a pair of parallel lines is cut by a transversal that is not perpendicular to the parallel lines:

• **Four of the angles are acute, and four are obtuse.**

• **All four acute angles are congruent.**

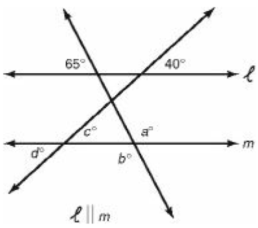
• **All four obtuse angles are congruent.**

• **The sum of the measures of any acute angle and any obtuse angle is 180°.**



> **TIP**
> The four acute angles all have the same measure, and the four obtuse angles all have the same measure.

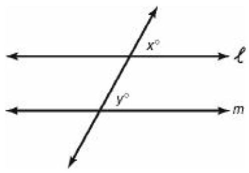
EXAMPLE 3: In the figure below, what is the value of $a + b + c + d$?



By KEY FACT G6, c and d each measure 40° and $65 + a = 180$. Therefore, $a = 115$. Since vertical angles have equal measures, b is also 115, and so

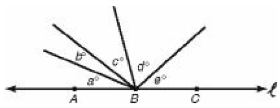
$$a + b + c + d = 115 + 115 + 40 + 40 = 310.$$

The converse of KEY FACT G6 is also true. If in the figure below, $x = y$, then lines ℓ and m are parallel.



Exercises

1. In the figure below, *A*, *B*, and *C* are on line ℓ. What is the average of *a*, *b*, *c*, *d*, and *e* ?



(A) 18

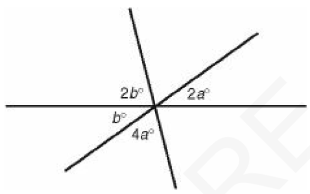
(B) 27

(C) 36

(D) 45

(E) 72

2. In the figure below, what is the value of *b* ?



(A) 9

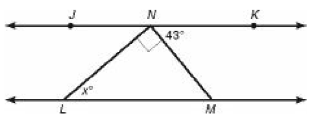
(B) 18

(C) 27

(D) 36

(E) 45

3. In the figure below $\overline{JK}$, is parallel $\overline{LM}$ to . What is the value of *x*?



(A) 37

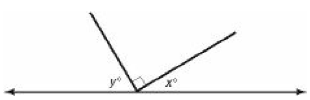
(B) 43

(C) 47

(D) 53

(E) 57

4. In the figure below, what is the value of *x* if the ratio of *y* to *x* is 7 to 3?



(A) 18

(B) 27

(C) 30

(D) 36

(E) 63

5. What is the measure of the angle formed by the minute and hour hands of a clock at 1:50?

(A) 90°

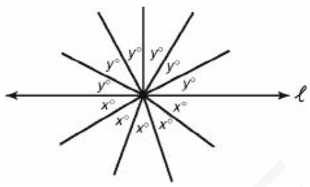
(B) 95°

(C) 105°

(D) 115°

(E) 120°

6. In the figure below, what is the value of $\frac{x+y}{x-y}$?



(A) 6

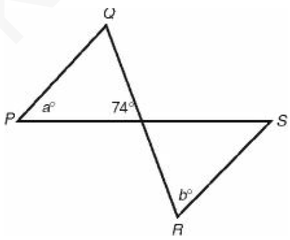
(B) 11

(C) 22

(D) 30

(E) 36

7. In the figure below, $\overline{PQ}$ is parallel to $\overline{RS}$. What is the value of $a + b$?



(A) 47

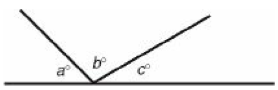
(B) 74

(C) 90

(D) 106

(E) It cannot be determined from the information given

8. In the figure below, $a : b = 3 : 5$ and $c : b = 2 : 1$. What is the measure of the largest angle?



Note: Figure not drawn to scale.

(A) 50°

(B) 90°

(C) 100°

(D) 120°

(E) 150°

9. *A*, *B*, and *C* are points on a line with *B* between *A* and *C*. Let *M* and *N* be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. If *AB* : *BC* = 3 : 1, what is *MN* : *BC*?

(A) $\frac{2}{3}$

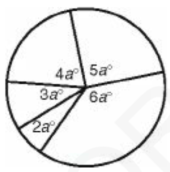
(B) $\frac{1}{1}$

(C) $\frac{3}{2}$

(D) $\frac{2}{1}$

(E) $\frac{3}{1}$

10. In the figure below, what is the average (arithmetic mean) of the measures of the five angles?



(A) 18°

(B) 36°

(C) 45°

(D) 72°

(E) It cannot be determined from the given information

ANSWERS EXPLAINED

Answer Key

1. **(C)**	5. **(D)**	8. **(C)**
2. **(D)**	6. **(B)**	9. **(D)**
3. **(C)**	7. **(D)**	10. **(D)**
4. **(B)**		

Solutions

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the "Tactics" chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(C)** Since $\angle ABC$ is a straight angle, by KEY FACT G2, the sum of *a*, *b*, *c*, *d*, and *e* is 180, and so their average is $\frac{180}{5} = 36$.

2. **(D)** Since vertical angles are congruent, the two unmarked angles are $2b$ and $4a$. Since the sum of all six angles is 360°:

$$360 = 4a + 2b + 2a + 4a + 2b + b = 10a + 5b$$

Since vertical angles are congruent, $b = 2a \Rightarrow 5b = 10a$. Hence:

$$360 = 10a + 5b = 10a + 10a = 20a \Rightarrow a = 18 \Rightarrow b = 36$$

3. **(C)** Let y be the degree measure of $\angle JNL$. Then by KEY FACT G2:

$$43 + 90 + y = 180 \Rightarrow 133 + y = 180 \Rightarrow y = 47$$

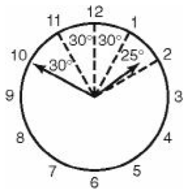
Since $\overline{LM}$ is parallel to $\overline{JK}$, by KEY FACT G6, $x = y \Rightarrow x = 47$.

4. **(B)** $x + y + 90 = 180 \Rightarrow x + y = 90$

Since the ratio of y to x is 7 to 3, $y = 7t$ and $x = 3t$. So:

$$7t + 3t = 90 \Rightarrow 10t = 90 \Rightarrow t = 9 \Rightarrow x = 3(9) = 27$$

5. **(D)** For problems such as this, always draw a picture. The measure of each of the 12 central angles from one number to the next is 30°. At 1:50, the minute hand is pointing at the 10, and the hour hand has gone $\frac{50}{60}=\frac{5}{6}$ of the way from the 1 to the 2. From the 10 to the 1 is 90° and from the 1 to the hour hand is $\frac{5}{6}(30°)=25°$, for a total of 90° + 25° = 115°.



6. **(B)** From the diagram, we see that $6y = 180$, which implies that $y = 30$ and that $5x = 180$, which implies that $x = 36$. So:

$$\frac{x+y}{x-y} = \frac{36+30}{36-30} = \frac{66}{6} = 11$$

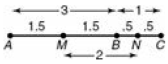
7. **(D)** Since $\overline{PQ}$ is parallel to $\overline{RS}$, $\angle R \cong \angle Q$ and so b = m$\angle Q$

$$m\angle Q + a + 74 = 180 \Rightarrow a + b + 74 = 180 \Rightarrow a + b = 106$$

8. **(C)** Since $a : b = 3 : 5$, $a = 3x$ and $b = 5x$. Since $c : b = 2 : 1$, $c = 2b = 10x$. Then:

$$3x + 5x + 10x = 180 \Rightarrow 18x = 180 \Rightarrow x = 10 \Rightarrow c = 10x = 100$$

9. **(D)** If a diagram is not provided on a geometry question, draw one. From the figure below, you can see that *MN* : *BC* = 2 : 1.



10. **(D)** The markings in the five angles are irrelevant. The sum of the measures of the five angles is 360°, and 360° ÷ 5 = 72°.

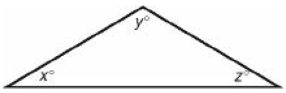
Triangles

Since more geometry questions on the Math 1 test concern triangles than any other topic, it is imperative that you learn all the KEY FACTS in this section.

SIDES AND ANGLES OF A TRIANGLE

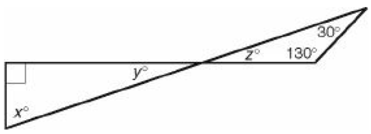
Key Fact H1

In any triangle, the sum of the measures of the three angles is 180º.

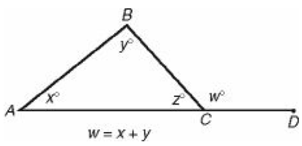


$x + y + z = 180$

EXAMPLE 1: To find the value of x in the figure below, use KEY FACT H1 twice. $130 + 30 + z = 180 \Rightarrow z = 20 \Rightarrow y = 20$, then $90 + x + 20 = 180 \Rightarrow x = 70$.



In the figure below, $\angle BCD$, which is formed by extending side $\overline{AC}$ of ΔABC, is called an ***exterior angle***.



Key Fact H2

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

In the diagram above, $w = x + y$.

Key Fact H3

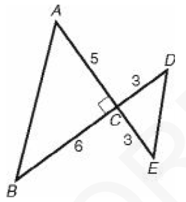
In any triangle:

- **The longest side is opposite the largest angle.**

- **The shortest side is opposite the smallest angle.**

- **Sides with equal lengths are opposite angles with equal measures (the angles opposite congruent sides are congruent).**

EXAMPLE 2: By using KEY FACT H3, we can draw the following conclusions concerning the angles in the figure below. In ΔCDE, since angles D and E are opposite congruent sides ($\overline{CE}$ and $\overline{CD}$, respectively), m$\angle D$ = m$\angle E$, angle C is the largest angle, so $\overline{AB}$ is the longest side. Specifically, $AB > 6$. Also, since $BC > AC$, m$\angle A$ > m$\angle B$.



- A triangle is called ***scalene*** if the three sides all have different lengths. Then by KEY FACT H3, the three angles all have different measures.

- A triangle is called ***isosceles*** if two sides are congruent. By KEY FACT H3, the two angles opposite the congruent sides are congruent.

- A triangle is called ***equilateral*** if all three sides are congruent. By KEY FACT H3, all three angles are congruent. Since the sum of the measures of the three angles is 180°, each angle measures 60°.

- ***Acute triangles*** are triangles in which all three angles are acute. An acute triangle could be scalene, isosceles, or equilateral.

- ***Obtuse triangles*** are triangles in which one angle is obtuse and two are acute. An obtuse triangle could be scalene or isosceles.

- ***Right triangles*** are triangles that have one right angle and two acute angles. A right triangle could be scalene or isosceles. The side opposite the 90° angle is called the ***hypotenuse***, and by KEY FACT H3, it is the longest side. The other two sides are called the ***legs***.

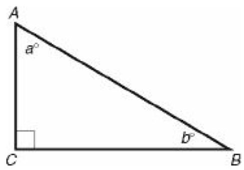
RIGHT TRIANGLES

If a and b are the measures in degrees, of the acute angles of a right triangle, then by KEY FACT H1, $90 + a + b = 180 \Rightarrow a + b = 90$.

Key Fact H4

In any right triangle, the sum of the measures of the two acute angles is 90°.

EXAMPLE 3: To find the average of a and b in ΔABC below, note that by KEY FACT H4, $a + b = 90$, so $\frac{a+b}{2} = \frac{90}{2} = 45$.



The most important fact concerning right triangles is the ***Pythagorean theorem***, which is given in the first line of KEY FACT H5. The second and third lines of KEY FACT H5 are important corollaries of the Pythagorean theorem.

Key Fact H5

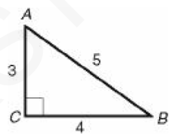
PYTHAGOREAN THEOREM AND COROLLARIES

Let a, b, and c be the lengths of the sides of ΔABC, with $a \leq b \geq c$.

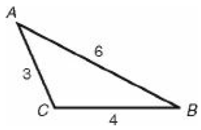
- **$a^2 + b^2 = c^2$ if and only if angle C is a right angle.**
- **$a^2 + b^2 < c^2$ if and only if angle C is obtuse.**
- **$a^2 + b^2 > c^2$, if and only if angle C is acute.**

TIP

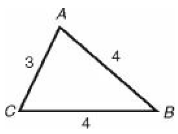
The Pythagorean theorem is probably the most important theorem you need to know. Be sure to review all of its uses.



$3^2 + 4^2 = 5^2$



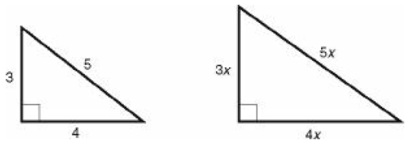
$3^2 + 4^2 < 6^2$



$3^2 + 4^2 > 4^2$

EXAMPLE 4: Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, a triangle whose sides have lengths 8, 15, and 17 is a right triangle. Since $10^2 + 15^2 = 100 + 225 = 325 < 19^2$, a triangle whose sides have lengths 10, 15, and 19 is not a right triangle-it is obtuse.

On the Math 1 test, the most common right triangles whose sides are integers are the 3-4-5 right triangle and its multiples.



Key Fact H6

For any positive number *x*, there is a right triangle whose sides are 3*x*, 4*x*, 5*x*.

For example:

$x = 1$	3, 4, 5
$x = 2$	6, 8, 10
$x = 3$	9, 12, 15
$x = 4$	12, 16, 20
$x = 5$	15, 20, 25
$x = 10$	30, 40, 50
$x = 50$	150, 200, 250
$x = 100$	300, 400, 500

Other right triangles with integer sides that you should recognize immediately are the ones whose sides are 5, 12, 13 and 8, 15, 17.

[redacted]

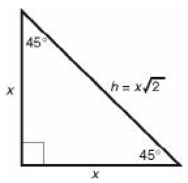
KEY FACT H6 applies even if x is not an integer. For example:

$x = 0.5$ 1.5, 2, 2.5
$x = \pi$ $3\pi, 4\pi, 5\pi$

SPECIAL RIGHT TRIANGLES

Let x be the length of each leg and let h be the length of the hypotenuse of an isosceles right triangle. By the Pythagorean theorem,

$$x^2 + x^2 = h^2 \Rightarrow 2x^2 = h^2 \Rightarrow h = \sqrt{2x^2} = x\sqrt{2}.$$

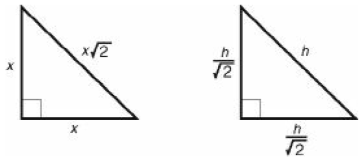


Key Fact H7

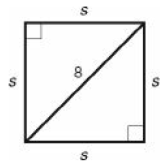
In a 45-45-90 right triangle, the sides are *x*, *x*, and *x* $\sqrt{2}$.

• **If you are given the length of a leg, multiply it by $\sqrt{2}$ to get the length of the hypotenuse.**

• **If you are given the length of the hypotenuse, divide it by $\sqrt{2}$ to get the length of each leg.**

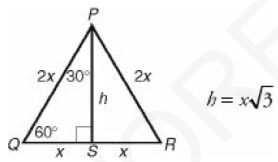


EXAMPLE 5: To find the area of a square whose diagonal is 8, note that the diagonal divides the square into two isosceles right triangles. So $s = \frac{8}{\sqrt{2}}$ and $A = s^2 = \left(\frac{8}{\sqrt{2}}\right)^2 = \frac{64}{2} = 32.$



Let 2*x* be the length of each side of equilateral Δ*PQR*, in which altitude $\overline{PS}$ has been drawn. Then Δ*PQS* is a 30-60-90 right triangle, and its sides are *x*, 2*x*, and *h*. By the Pythagorean theorem,

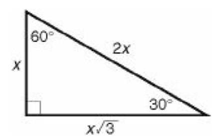
$$x^2 + h^2 = (2x)^2 = 4x^2 \Rightarrow h^2 = 3x^2 \Rightarrow h = \sqrt{3x^2} = x\sqrt{3}.$$



Key Fact H8

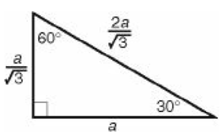
In a 30-60-90 right triangle, the sides are *x*, *x* $\sqrt{3}$, and 2*x*.
If you know the length of the shorter leg (*x*):

- **Multiply it by $\sqrt{3}$to get the length of the longer leg.**
- **Multiply it by 2 to get the length of the hypotenuse.**



If you know the length of the longer leg (*a*):

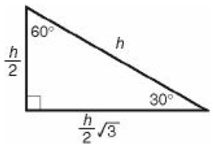
- **Divide it by $\sqrt{3}$ to get the length of the shorter leg.**
- **Multiply the length of the shorter leg by 2 to get the length of the hypotenuse.**



If you know the length of the hypotenuse (*h*):

- **Divide it by 2 to get the length of the shorter leg.**

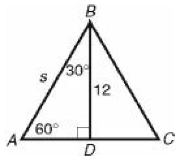
• **Multiply the length of the shorter leg by $\sqrt{3}$ to get the length of the longer leg.**



PERIMETER AND AREA

The ***perimeter*** of a triangle is the sum of the lengths of the three sides.

EXAMPLE 6: To find the perimeter of an equilateral triangle whose height is 12, note that the height divides the triangle into two 30-60-90 right triangles. In the figure below, by KEY FACT H8,



$$AD = \frac{12}{\sqrt{3}} \text{ and } s = 2\left(\frac{12}{\sqrt{3}}\right)$$

$$s = 2\left(\frac{12}{\sqrt{3}}\right) = \frac{24}{\sqrt{3}} = \frac{24}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3}$$

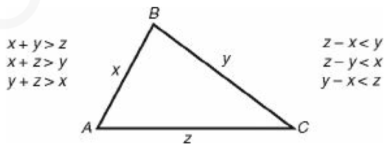
So the perimeter is $3s = 24\sqrt{3}$.

Key Fact H9

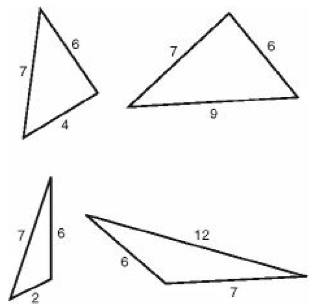
TRIANGLE INEQUALITY

• **The sum of the lengths of any two sides of a triangle is greater than the length of the third side.**

• **The difference of the lengths of any two sides of a triangle is less than the length of the third side.**



EXAMPLE 7: A teacher asked her class to draw triangles in which the lengths of two of the sides were 6 inches and 7 inches and the length of the third side was also a whole number of inches. To determine how many different triangles the class could draw, note that if x represents the length of the third side, then by KEY FACT H9, $6 + 7 > x$, so $x < 13$. Also, $7 – 6 < x$, so $x > 1$. So x could have 11 different values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. The perimeters of the triangles are the integers from 15 to 25. The following diagram illustrates four of the triangles the class could have drawn.



Frequently, questions on the Math 1 test require you to calculate the ***area*** of a triangle.

Key Fact H10

The area of a triangle is given by $A = \frac{1}{2}$ *bh*, where *b* and *h* are the lengths of the base and height, respectively.

(1) *Any* side of the triangle can be taken as the base.

(2) The ***height*** (which is also called an ***altitude***) is a line segment drawn perpendicular to the base from the opposite vertex.

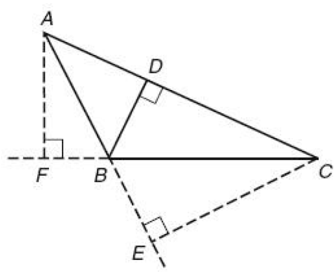
(3) In a right triangle, either leg can be the base and the other the height.

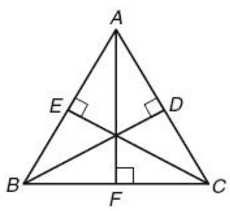
(4) If one endpoint of the base is the vertex of an obtuse angle, then the height will be outside the triangle.

(5) In each figure at the top of this page

- If $\overline{AC}$ is the base, $\overline{BD}$ is the height.
- If $\overline{AB}$ is the base, $\overline{CE}$ is the height.
- If $\overline{BC}$ is the base, $\overline{AF}$ is the height.

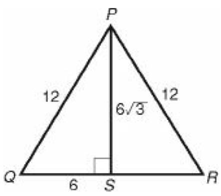
> **TIP**
>
> The height can be outside the triangle. See $\overline{AF}$ in the diagram shown below.





EXAMPLE 8: To find the area of equilateral triangle *PQR*, below, whose sides are 12, draw in altitude $\overline{PS}$. ΔPQS is a 30-60-90 right triangle, and so by KEY FACT H8, *QS* = 6 and *PS* =$6\sqrt{3}$. Finally, the area of

$$\Delta PQS = \frac{1}{2}(12)(6\sqrt{3}) = 36\sqrt{3}.$$



Replacing 12 by *s* in Example 8 yields a useful formula.

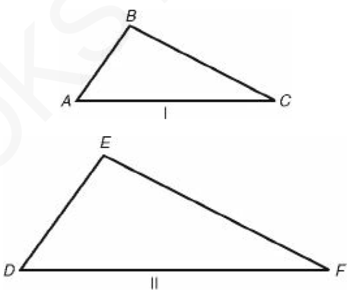
Key Fact H11

if *A* represents the area of an equilateral triangle with side *s*, then $A = \frac{s^2\sqrt{3}}{4}$**.**

Learning this formula for the area of an equilateral triangle can save you time.

SIMILAR TRIANGLES

Two triangles, such as triangle I and triangle II in the figure below, that have the same shape but not necessarily the same size are said to be ***similar***.



The following KEY FACT makes this intuitive definition mathematically precise.

Key Fact H12

Two triangles are *similar* provided the following two conditions are satisfied:

- **The three angles in the first triangle have the same measures as the three angles in the second triangle.**

  $m\angle A = m\angle D \qquad m\angle B = m\angle E \qquad m\angle C = m\angle F$

- **The lengths of the corresponding sides of the two triangles are in proportion.**

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

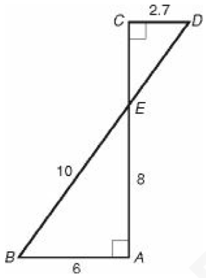
Corresponding sides are sides that are opposite angles of the same measure.

An important theorem in geometry states that if the first condition in KEY FACT H12 is satisfied, then the second condition is automatically satisfied. Therefore, to show that two triangles are similar, it is sufficient to show that their angles have the same measures. Furthermore, if the measures of two angles of one triangle are equal to the measures of two angles of a second triangle, then the measures of the third angles are also equal. KEY FACT H13 is an immediate consequence of the results in this paragraph.

Key Fact H13

If the measures of two angles of one triangle are equal to the measures of two angles of a second triangle, then the triangles are similar.

EXAMPLE 9: Find the length of $\overline{ED}$ in the diagram below.

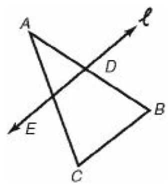


First note that since vertical angles have equal measures, m∠*AEB* = m∠*DEC* and, of course, m∠*A* = m∠*C*. So the measures of two angles of Δ*ABE* are equal to the measures of two angles of Δ*CDE*. By KEY FACT H13, the two triangles are similar. So, by KEY FACT H12, corresponding sides of the triangles are in proportion. Therefore:

$$\frac{AB}{DC} = \frac{BE}{ED} \Rightarrow \frac{6}{2.7} = \frac{10}{ED}$$

6(*ED*) = 27, and so cross-multiplying, we get $ED = \frac{27}{6} = 4.5$

In Δ*ABC* below, line ℓ is parallel to side $\overline{BC}$ and intersects sides $\overline{AB}$ and $\overline{AC}$ at *D* and *E*, respectively. Then by KEY FACT G6, m∠*AED* = m∠*C* and m∠*ADE* = m∠*B*. So by KEY FACT H13, Δ*AED* is similar to Δ*ACB*.



Key Fact H14

A line that intersects two sides of a triangle and is parallel to the third side creates a smaller triangle that is similar to the original one.

If two triangles are similar, the common ratio of their corresponding sides is called the ***ratio of similitude***.

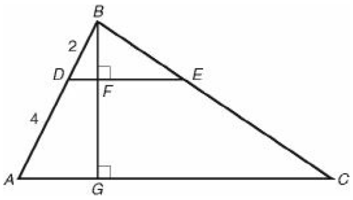
Key Fact H15

If two triangles are similar, and if *k* is the ratio of similitude:

• The ratio of all their linear measurements is *k*.

• The ratio of their areas is *k*2.

In the figure below, $\overline{DE} \parallel \overline{AC}$. By KEY FACT H14, ΔABC and ΔDBE are similar.



Then $\overline{AB}$ and $\overline{DB}$ are corresponding sides. Since $AB = 6$ and $DB = 2$, the ratio of similitude is $\frac{6}{2} = \frac{3}{1}$. Therefore:

• All the sides are in the ratio of $\frac{3}{1}$: $BC = \mathbf{3} \times BE$ and $AC = \mathbf{3} \times DE$

• The altitudes are in the ratio of $\frac{3}{1}$: $BG = \mathbf{3} \times BF$

• The perimeters are in the ratio of : $\frac{3}{1}$

$$\text{perimeter of } \Delta ABC = \mathbf{3} \times (\text{perimeter of } \Delta DBE)$$

• The areas are in the ratio of $\frac{9}{1}$: area of $\Delta ABC = \mathbf{9} \times$ (area of ΔDBE)

Exercises

1. In ΔABC, the measures of the exterior angles at vertices A and B are 145° and 125°, respectively. Which of the following statements about ΔABC must be true?

I. The triangle is a right triangle.

II. The triangle is an isosceles triangle.

III. The triangle is a scalene triangle.

(A) None

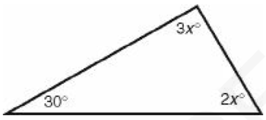
(B) I only

(C) III only

(D) I and II only

(E) I and III only

2. In the triangle below, what is the value of x?



(A) 20

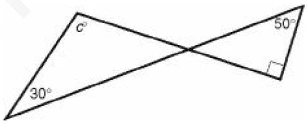
(B) 30

(C) 40

(D) 50

(E) 60

3. In the figure below, what is the value of c ?



(A) 100

(B) 110

(C) 120

(D) 130

(E) 140

4. The lengths of two sides of a right triangle are 15 and 17. Which of the following *could* be the length of the third side?

I. 8

II. 19

III. $\sqrt{514}$

(A) I only

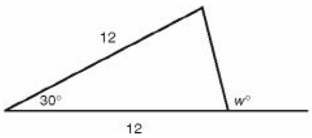
(B) II only

(C) I and II

(D) I and III

(E) I, II, and III

5. What is the value of w in the figure below?



(A) 60

(B) 90

(C) 105

(D) 120

(E) 150

6. What is the area of an equilateral triangle whose altitude is 6?

(A) 18

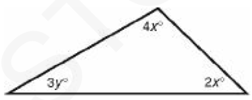
(B) $12\sqrt{3}$

(C) $18\sqrt{3}$

(D) 36

(E) $24\sqrt{3}$

7. Which of the following expresses a true relationship between x and y in the figure below?



(A) $y = 60 - 2x$

(B) $y = 2x$

(C) $x + y = 90$

(D) $y = 180 - 3x$

(E) $x = 90 - 3y$

8. What is the value of AD in the triangle below?

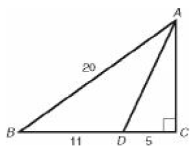
(A) $5\sqrt{2}$

(B) 10

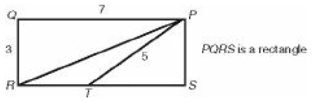
(C) 11

(D) 13

(E) $12\sqrt{2}$



Questions 9 and 10 refer to the following figure.



9. What is the area of ΔPRT?

(A) 3

(B) 4.5

(C) 6

(D) 7.5

(E) 10

10. What is the perimeter of ΔPRT?

(A) $8 + \sqrt{41}$

(B) $8 + \sqrt{58}$

(C) 16

(D) 17

(E) 18

11. If the measures of the angles of a triangle are in the ratio of 1 : 2 : 3, what is the ratio of the length of the smallest side of the triangle to the length of the longest side of the triangle?

(A) 1 : 2

(B) 1 : 3

(C) 1 : 5

(D) 2 : 3

(E) 2 : 5

12. If the difference between the measures of the two acute angles of a right triangle is 20°, what is the measure, in degrees, of the smaller one?

(A) 20°

(B) 35°

(C) 45°

(D) 55°

(E) 80°

13. If one side of ΔABC is 5, what is the smallest integer that could be the perimeter of the triangle?

(A) 8

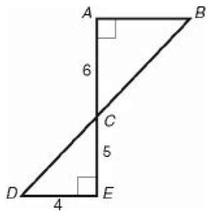
(B) 10

(C) 11

(D) 12

(E) 15

14. In the figure below, what is the length of $\overline{BD}$?



(A) 14.00

(B) 14.09

(C) 14.21

(D) 15.62

(E) 16.00

15. The lengths of the sides of a triangle are 3, 5, and *x*. How many possible values of *x* are there, if *x* must be an integer?

(A) 1

(B) 5

(C) 7

(D) 8

(E) Infinitely many

ANSWERS EXPLAINED

Answer Key

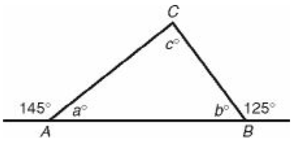
1. **(E)**	6. **(B)**	11. **(A)**
2. **(B)**	7. **(A)**	12. **(B)**
3. **(B)**	8. **(D)**	13. **(C)**
4. **(D)**	9. **(B)**	14. **(B)**
5. **(C)**	10. **(B)**	15. **(B)**

Solutions

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the “Tactics” chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

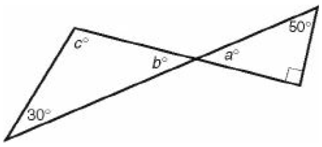
1. **(E)** First, draw ΔABC.



By KEY FACT G2, $a + 145 = 180$ and $b + 125 = 180 \Rightarrow a = 35$ and $b = 55$. Then by KEY FACT H1, $35 + 55 + c = 180 \Rightarrow c = 90$ (I is true). Since the measures of all three angles are different, by KEY FACT H3, the lengths of all three sides are different (II is false and III is true). Only I and III are true.

2. **(B)** $2x+3x+30=180 \Rightarrow 5x+30=180 \Rightarrow 5x=150 \Rightarrow x=30$

3. **(B)** In the figure below, $50 + 90 + a = 180 \Rightarrow a = 40$. Since vertical angles are congruent, b = 40. So 40 + 30 + C = 180 ⇒ C = 110.



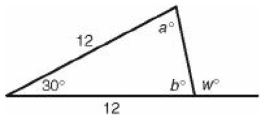
4. **(D)** If the triangle were not required to be right, by KEY FACT H9 *any* number greater than 2 and less than 32 could be the length of the third side. For a right triangle, though, there are only *two* possibilities:

- 17 is the length of the hypotenuse, and the legs have lengths 15 and 8. I is true. (If you did not recognize the 8-15-17 triangle, use the Pythagorean theorem: solve the equation $15^2 + x^2 = 17^2$.)
- 15 and 17 are the lengths of the two legs. Use the Pythagorean theorem to find the hypotenuse:

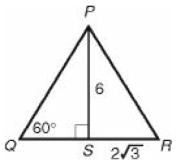
$15^2 + 17^2 = c^2 \Rightarrow c^2 = 225 + 289 = 514 \Rightarrow c = \sqrt{514}$. III is true.

- Since $15^2 + 17^2 \neq 19^2$, a 15-17-19 triangle is *not* a right triangle. II is false. Only I and III are true.

5. **(C)** Since the triangle is isosceles, in the figure below, *a* and *b* are equal. Since $a + b + 30 = 180$, *a* and *b* are each 75. Finally, $w = 180 - 75 = 105$.



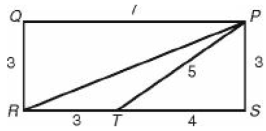
6. **(B)** Draw altitude $\overline{PS}$ in equilateral ΔPQR. By KEY FACT H8, $SR = \frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$. Since, in ΔPQR, SR is one-half the base, the area of the triangle is $\left(\frac{1}{2}b\right)h = 2\sqrt{3} \times 6 = 12\sqrt{3}$.



7. **(A)** $2x + 4x + 3y = 180 \Rightarrow 6x + 3y = 180 \Rightarrow 3y = 180 - 6x \Rightarrow y = 60 - 2x$

8. **(D)** Use the Pythagorean theorem twice, unless you recognize the common right triangles in this figure (*which you should*). Since $AB = 20$ and $BC = 16$, ΔABC is a $3x$-$4x$-$5x$ right triangle with $x = 4$. So $AC = 12$, and ΔACD is a right triangle whose legs are 5 and 12. Its hypotenuse, AD, is therefore 13.

9. **(B)** Since ΔPST is a right triangle whose hypotenuse is 5 and one of whose legs is 3, the other leg, $\overline{ST}$, is 4. Since $SR = PQ = 7$, $TR = 3$. Now, ΔPRT has a base of 3 ($\overline{TR}$) and a height of 3 ($\overline{PS}$), and so its area is $\frac{1}{2}(3)(3) = 4.5$.



10. **(B)** In Δ*PRT*, we already have that *PT* = 5 and *TR* = 3; we need only find *PR*, which is the hypotenuse of Δ*PQR*. By the Pythagorean theorem:

$$3^2 + 7^2 = (PR)^2 \Rightarrow (PR)^2 = 9 + 49 = 58 \Rightarrow PR = \sqrt{58}.$$

So the perimeter is $3 + 5 + \sqrt{58} = 8 + \sqrt{58}$.

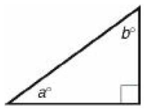
11. **(A)** If the measures of the three angles are in the ratio of 1 : 2 : 3, then for some number *x*, their values are *x*, 2*x*, and 3*x*.

$$x + 2x + 3x = 180° \Rightarrow 6x = 180° \Rightarrow x = 30°$$

So the triangle is a 30-60-90 right triangle, and for some number *a*, the lengths of the sides are *a*, $a\sqrt{3}$, and 2*a*. The ratio of the length of the smallest side to the length of the largest side is *a* : 2*a* = 1 : 2.

12. **(B)** Draw a picture, label it, and then write the equations:

$$a + b = 90 \text{ and } a - b = 20$$



By adding the equations, we get $2a = 110$. So $a = 55$ and $b = 90 - 55 = 35$. The measure of the smallest angle is 35°.

13. **(C)** If *x* and *y* are the lengths of the other two sides, then by KEY FACT H9, $x+y>5 \Rightarrow x+y+5>10$. So the smallest integer that the perimeter could be is 11. (In fact, there is an isosceles triangle whose sides are 3, 3, and 5.)

14. **(B)** The two vertical angles at *C* are congruent and $\angle A \cong \angle E$ since they are both right angles. Therefore, by KEY FACT H13, ΔCAB and ΔCED are similar. So their sides are in proportion, with a ratio of similitude of $\frac{AC}{CE} = \frac{6}{5} = 1.2$. By the Pythagorean theorem, $(CD)^2 = 4^2 + 5^2 = 16 + 25 = 41$. So $CD = \sqrt{41}$. $\frac{BC}{CD} = 1.2 \Rightarrow BC = 1.2\sqrt{41}$. Finally, $BD = \sqrt{41} + 1.2\sqrt{41} = 14.09$.

15. **(B)** By the triangle inequality (KEY FACT H9),

$$(5 - 3) < x < (5 + 3) \Rightarrow 2 < x < 8.$$

So there are 5 possible integer values of x: 3, 4, 5, 6, 7.

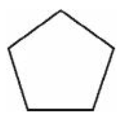
Quadrilaterals and Other Polygons

A ***polygon*** is a closed geometric figure made up of line segments. The line segments are called ***sides***, and the endpoints of the line segments are called ***vertices*** (each one is called a ***vertex***). Line segments inside the polygon drawn from one vertex to another are called ***diagonals***.

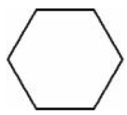
Three-sided polygons, called triangles, were discussed in Chapter Although in this section our main focus will be on four-sided polygons, which are called ***quadrilaterals***, we will discuss other polygons as well. There are special names for many polygons with more than four sides. The ones you need to know for the Math 1 test are given in the following chart.

Number of Sides	Name	Number of Sides	Name
5	Pentagon	8	Octagon
6	Hexagon	10	Decagon

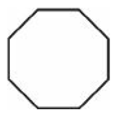
A ***regular polygon*** is a polygon in which all the sides have the same length and all the angles have the same measure. A regular three-sided polygon is an equilateral triangle, and, as we shall see, a regular quadrilateral is a square. Pictured below are a regular pentagon, regular hexagon, and regular octagon.



pentagon



hexagon

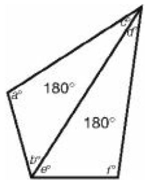


octagon

In a *regular* polygon, all the angles are congruent and all the sides are congruent.

THE ANGLES OF A POLYGON

A diagonal of a quadrilateral divides it into two triangles. Since the sum of the measures of the three angles in each of the triangles is 180°, the sum of the measures of the angles in the quadrilateral is 360°.

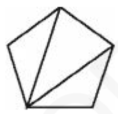


$$a + b + c = 180 \text{ and } d + e + f = 180$$
$$a + (b + e) + (c + d) + f = 360$$

Key Fact I1

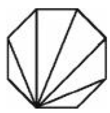
In any quadrilateral, the sum of the measures of the four angles is 360°.

Similarly, any polygon can be divided into triangles by drawing in all of the diagonals emanating from one vertex.



pentagon

hexagon



octagon

Notice that a five-sided polygon can be divided into three triangles, and a six-sided polygon can be divided into four triangles. In general, an n-sided polygon can be divided into $(n – 2)$ triangles, which leads to KEY FACT I2.

Key Fact I2

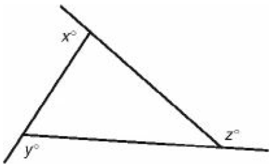
The sum of the measures of the n angles in a polygon with n sides is $(n – 2) \times 180°$.

EXAMPLE 1: To find the measure of each angle of a regular octagon, first use KEY FACT I2 to get that the sum of all eight angles is $(8 – 2) \times 180° = 6 \times 180° = 1,080°$. Then since in a regular octagon all eight angles have the same measure, the measure of each one is $1,080° \div 8 = 135°$.

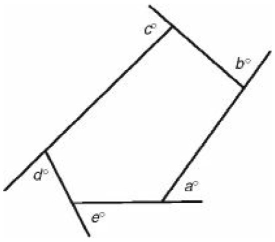
An ***exterior angle*** of a polygon is formed by extending a side. Surprisingly, in all polygons, the sum of the measures of the exterior angles is the same.

Key Fact I3

In any polygon, the sum of the measures of the exterior angles, taking one at each vertex, is 360°.



$x + y + z = 360$



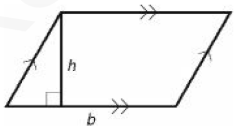
$a + b + c + d + e = 360$

EXAMPLE 2: KEY FACT I3 gives us an alternative method of calculating the measure of each angle in a regular polygon. In Example 1 we used KEY FACT I2 to find the measure of each angle in a regular octagon. By KEY FACT I3, the sum of the measures of the eight exterior angles of any octagon is 360°. As a result, in a regular octagon, the measure of each exterior angle is 360° ÷ 8 = 45°. Therefore, the measure of each interior angle is 180° – 45° = 135°.

SPECIAL QUADRILATERALS

We will now define five special quadrilaterals and review the important properties of each one.

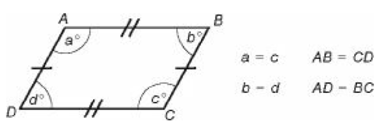
A ***parallelogram*** is a quadrilateral in which both pairs of opposite sides are parallel. Any side of a parallelogram can be its ***base***, and a line segment drawn from a vertex perpendicular to the opposite base is called the ***height***.



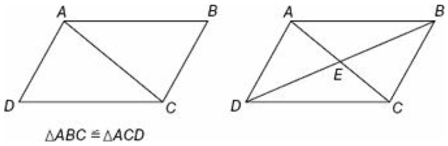
Key Fact I4

Parallelograms have the following properties illustrated in the figures below:

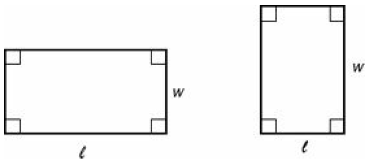
- **Opposite sides are parallel:** $\overline{AB} \parallel \overline{CD}$ **and** $\overline{AD} \parallel \overline{BC}$.
- **Opposite sides are congruent:** $\overline{AB} \cong \overline{CD}$ **and** $\overline{AD} \cong \overline{BC}$.
- **Opposite angles are congruent:** $\angle A \cong \angle C$ **and** $\angle B \cong \angle D$.
- **The sum of the measures of any pair of consecutive angles is 180°. For example, $a + b = 180$ and $b + c = 180$.**



- **A diagonal divides the parallelogram into two congruent triangles:** $\Delta ABC \cong \Delta ACD$.
- **The two diagonals bisect each other:** $AE = EC$ **and** $BE = ED$**.**



A ***rectangle*** is a parallelogram in which all four angles are right angles. Two adjacent sides of a rectangle are usually called the ***length*** (ℓ) and the ***width*** (w). Note that the length is not necessarily greater than the width.



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A rectangle is a parallelogram.

Key Fact I5

Since a rectangle is a parallelogram, all of the properties listed in KEY FACT I4 hold for rectangles. In addition:

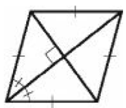
- **The measure of each angle in a rectangle is 90°.**
- **The diagonals of a rectangle have the same length: *AC* = *BD*.**

A ***rhombus*** is a parallelogram in which all four sides have the same length.

Key Fact I6

Since a rhombus is a parallelogram, all of the properties listed in KEY FACT I4 hold for rhombuses. In addition:

- **The length of each side of a rhombus is the same.**
- **The two diagonals of a rhombus are perpendicular.**
- **The diagonals of a rhombus are angle bisectors.**



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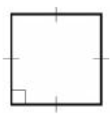
A rhombus is a parallelogram.

A ***square*** is a rectangle in which all four sides have the same length. So a square is both a rectangle and a rhombus.

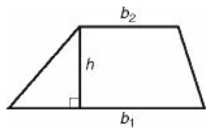


Key Fact I7

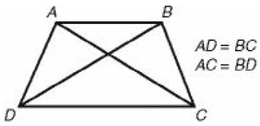
Since a square is a rectangle and a rhombus, all of the properties listed in KEY FACTS I4, I5, and I6 hold for squares.

A square is a special parallelogram, which is both a rectangle and a rhombus.

A ***trapezoid*** is a quadrilateral in which exactly one pair of opposite sides is parallel. The parallel sides are called the ***bases*** of the trapezoid, and the distance between the two bases is called the ***height***. If the two nonparallel sides are congruent, the trapezoid is called ***isosceles*** and, in that case only, the diagonals are congruent.



trapezoid



isosceles trapezoid

PERIMETER AND AREA OF QUADRILATERALS

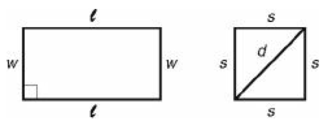
The ***perimeter*** (P) of a polygon is the sum of the lengths of all of its sides. The ***area*** (A) of a polygon is the amount of space it encloses (measured in square units). The perimeter and area formulas you need to know for the Math 1 test are given in KEY FACTS I8 and I9.

Key Fact I8

PERIMETERS

• **For a rectangle:** $P = 2(\ell + w)$

• **For a square:** $P = 4s$



Key Fact I9

AREAS

• **For a parallelogram:** $A = bh$

• **For a rectangle:** $A = \ell w$

• **For a square:** $A = s^2$ **or** $A = \frac{1}{2}d^2$

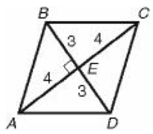
• **For a trapezoid:** $A = \frac{1}{2}(b_1 + b_2)h$

TIP

Here's a useful alternate formula for the area of a square:

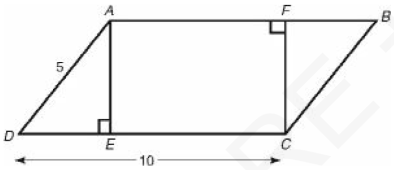
If d is the diagonal of a square, then the area of the square is $\frac{d^2}{2}$.

EXAMPLE 3: What are the perimeter and area of a rhombus whose diagonals are 6 and 8? First draw and label a rhombus.



Since the diagonals bisect each other, $BE = ED = 3$ and $AE = EC = 4$. Also, since the diagonals of a rhombus are perpendicular, $\angle BEA$ is a right angle and $\angle BEA$ is a 3-4-5 right triangle. So $AB = 5$ and the perimeter of the rhombus is $4 \times 5 = 20$. The easiest way to calculate the area of the rhombus is to recognize that it is the sum of the areas of four 3-4-5 right triangles. Since each triangle has an area of $\frac{1}{2}(3)(4) = 6$, the area of the rhombus is $4 \times 6 = 24$.

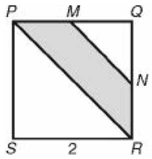
EXAMPLE 4: In the figure below, the area of parallelogram *ABCD* is 40. What are the areas of rectangle *AFCE*, trapezoid *AFCD*, and triangle *BCF* ?



Since the base of parallelogram *ABCD* is 10 and its area is 40, its height, *AE*, must be 4. Then ΔAED must be a 3-4-5 right triangle with $DE = 3$, which implies that $EC = 7$. So the area of rectangle *AFCE* is $7 \times 4 = 28$; the area of trapezoid *AFCD* is $\frac{1}{2}(10 + 7)(4) = 34$; and the area of each small triangle is $\frac{1}{2}(3)(4) = 6$.

Exercises

Questions 1 and 2 refer to the following figure, in which *M* and *N* are midpoints of two of the sides of square *PQRS*.



1. What is the perimeter of trapezoid *PMNR*?

   (A) 3

   (B) $2 + 3\sqrt{2}$

   (C) $3 + \sqrt{2}$

   (D) 5

   (E) 8

2. What is the area of trapezoid *PMNR*?

   (A) 1.5

   (B) 1.75

   (C) 3

   (D) $2\sqrt{2}$

   (E) $3\sqrt{2}$

3. The length of a rectangle is 5 more than the side of a square, and the width of the rectangle is 5 less than the side of the square. If the area of the square is 45, what is the area of the rectangle?

   (A) 20

   (B) 25

   (C) 45

   (D) 50

   (E) 70

4. In rhombus *PQRS*, the ratio of $m\angle P$ to $m\angle Q$ is 1 to 5 and $PQ = 6$. What is the area of the rhombus?

   (A) 6

   (B) 12

   (C) 18

   (D) 24

   (E) 30

5. If the length of a rectangle is 5 times its width and if its area is 180, what is its perimeter?

   (A) 6

   (B) 36

   (C) 60

   (D) 72

(E) 144

6. What is the average (arithmetic mean) of the measures of all the interior angles in a decagon?

(A) 18

(B) 36

(C) 72

(D) 90

(E) 144

7. If the interior angles of a pentagon are in the ratio of 2 : 3 : 3 : 5 : 5, what is the measure of the smallest angle?

(A) 20°

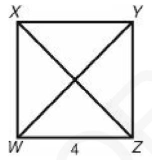
(B) 40°

(C) 60°

(D) 80°

(E) 90°

8. In the figure below, the two diagonals divide square *WXYZ* into four small triangles. What is the sum of the perimeters of those four triangles?



(A) $4 + 4\sqrt{2}$

(B) $16 + 8\sqrt{2}$

(C) $16 + 16\sqrt{2}$

(D) 32

(E) 48

9. How many sides does a polygon have if the measure of each interior angle is 9 times the measure of each exterior angle?

(A) 8

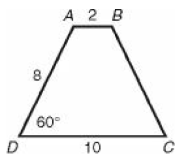
(B) 10

(C) 12

(D) 18

(E) 20

10. What is the area of trapezoid *ABCD* in the figure below?



(A) 24

(B) 28

(C) $24\sqrt{3}$

(D) $28\sqrt{3}$

(E) $48\sqrt{3}$

11. What is the area of a regular hexagon whose sides are 4?

(A) 24

(B) 36

(C) $24\sqrt{3}$

(D)$36\sqrt{3}$

(E)$48\sqrt{3}$

12. The area of a regular octagon whose sides are 2 can be expressed as $a+b\sqrt{2}$. What is the value of $a + b$?

(A) 8

(B) 12

(C) 16

(D) 18

(E) 24

ANSWERS EXPLAINED

Answer Key

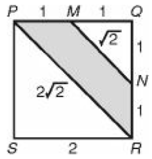
1. **(B)**	5. **(D)**	9. **(E)**
2. **(A)**	6. **(E)**	10. **(C)**
3. **(A)**	7. **(C)**	11. **(C)**
4. **(C)**	8. **(C)**	12. **(C)**

Solutions

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the “Tactics” chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

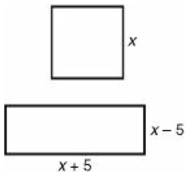
See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(B)** Since *M* and *N* are midpoints of sides of length 2, $PM = MQ = QN = NR = 1$. Since $\overline{MN}$ is the hypotenuse of an isosceles right triangle whose legs are 1, $MN = \sqrt{2}$. Similarly, $PR = 2\sqrt{2}$, since it is the hypotenuse of an isosceles right triangle whose legs are 2. So the perimeter of trapezoid *PMNR* is $1+\sqrt{2}+1+2\sqrt{2}=2+3\sqrt{2}$.

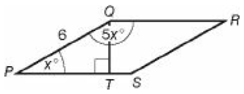


2. **(A)** Even if you know the formula for the area of a trapezoid (and you should), the best way to proceed is to subtract the areas of the two white triangles from 4, the area of the square. The area of $\Delta PSR = \frac{1}{2}(2)(2) = 2$, and the area of ΔMQN is $\frac{1}{2}(1)(1) = 0.5$ Therefore, the area of the shaded region is 4 – 2 – 0.5 = 1.5.

3. **(A)** Let x represent the side of the square. Then the dimensions of the rectangle are $(x + 5)$ and $(x - 5)$, and its area is $(x + 5)(x - 5) = x^2 - 25$. Since the area of the square is 45, $x^2 = 45$, and so $x^2 - 25 = 20$.



4. **(C)** Draw and label the rhombus.

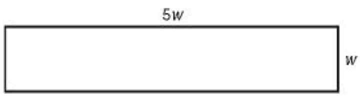


Since angles *P* and *Q* are supplementary:

$$m\angle P + m\angle Q = 180° \Rightarrow 5x + x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30$$

Then since ΔPQT is a 30-60-90 right triangle, $QT = \frac{6}{2} = 3$. Finally, the area of rhombus *PQRS* = *bh* = (*PS*)(*QT*) = (6)(3) = 18.

5. **(D)** Draw a diagram and label it.



Since the area is 180, we have $180 = 5w^2 \Rightarrow w^2 = 36 \Rightarrow w = 6$. So the width is 6, the length is $5 \times 6 = 30$, and the perimeter is $2(6 + 30) = 2(36) = 72$.

6. **(E)** By KEY FACT I2, the sum of the measures of the 10 interior angles in a decagon is $(10 - 2) \times 180° = 8(180°) = 1{,}440°$. So the average of their measures is $1{,}440° \div 10 = 144°$.

7. **(C)** By KEY FACT I2, the sum of the angles of a pentagon is

$$(5-2)\times 180° = 3\times 180° = 540°$$

Let the degree measures of the five angles be 2*x*, 3*x*, 3*x*, 5*x*, and 5*x*. Then

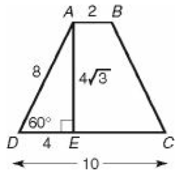
$$540 = 2x + 3x + 3x + 5x + 5x = 18x \Rightarrow$$
$$x = (540 \div 18) = 30$$

The degree measure of the smallest angle is 2*x*, and $2\times 30° = 60°$.

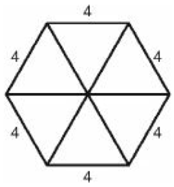
8. **(C)** Since the diagonals of a square are perpendicular, congruent, and bisect each other, each small triangle is a 45-45-90 right triangle whose hypotenuse is 4. Therefore, the legs are each $\frac{4}{\sqrt{2}} = 2\sqrt{2}$. So the perimeter of each small triangle is $4 + 2\sqrt{2} + 2\sqrt{2} = 4 + 4\sqrt{2}$, and the sum of the perimeters is $4(4 + 4\sqrt{2}) = 16 + 16\sqrt{2}$.

9. **(E)** The sum of an interior and exterior angle is 180°. So $180 = 9x + x = 10x \Rightarrow x = 18$. Since the sum of all the exterior angles is 360, there are $360 \div 18 = 20$ exterior angles, 20 interior angles, and 20 sides.

10. **(C)** Draw in height $\overline{AE}$. Then ΔAED is a 30-60-90 right triangle whose hypotenuse is 8. So $DE = 4$ and $AE = 4\sqrt{3}$. Then the area of trapezoid $ABCD$ is $\frac{1}{2}(10+2)\left(4\sqrt{3}\right) = 6\left(4\sqrt{3}\right) = 24\sqrt{3}$.

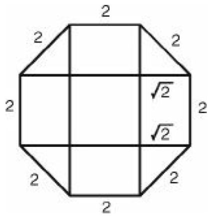


11. **(C)** Since you do not know a formula for the area of a hexagon, you have to divide the hexagon into manageable pieces. There are many ways to do this. One way is to divide it into two trapezoids; another is to divide it into a rectangle and two triangles. The simplest way, though, is to divide it into six equilateral triangles.



By KEY FACT H11, the area of an equilateral triangle whose sides are 4 is $\frac{4^2\sqrt{3}}{4} = \frac{16\sqrt{3}}{4} = 4\sqrt{3}$. So the area of the hexagon is $6\left(4\sqrt{3}\right) = 24\sqrt{3}$.

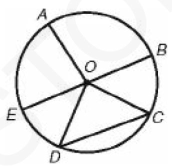
12. **(C)** Since you do not know a formula for the area of an octagon, you have to divide the octagon into manageable pieces. There are many ways to do that. One way is to draw four diagonals that divide the octagon into four isosceles right triangles, four rectangles, and a square.



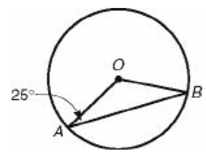
The legs of the triangles are each $\sqrt{2}$. Therefore, each triangle has area 1, each rectangle has area $2\sqrt{2}$, and the square has area 4. So the total area is $4(1) + 4(2\sqrt{2}) + 4 = 8 + 8\sqrt{2}$. So $a = 8$, $b = 8$, and $a + b = 16$.

Circles

A ***circle*** consists of all the points that are the same distance from one fixed point called the ***center***. That distance is called the ***radius*** of the circle. The figure below is a circle of radius 1 unit whose center is at the point O. A, B, C, D, and E, which are each 1 unit from O, are all points on circle O. The word radius is also used to represent any of the line segments joining the center and a point on the circle. The plural of *radius* is *radii*. In circle O below, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OE}$ are all radii. If a circle has radius r, each of the radii is r units long. A point is inside a circle if the distance from the center to that point is less than the radius. A point is outside a circle if the distance from the center to that point is greater than the radius.



EXAMPLE 1: In the figure below, O is the center of the circle. To find m$\angle B$ and m$\angle O$, observe that since $\overline{OA}$ and $\overline{OB}$ are radii, $OA = OB$ and ΔAOB is isosceles. So m$\angle B = 25°$ and m$\angle O = 180° - (25° + 25°) = 130°$.



Key Fact J1

Any triangle formed by connecting the endpoints of two radii is isosceles.

A ***chord*** of a circle is a line segment that has both endpoints on the circle. In the figure at the beginning of this chapter, $\overline{CD}$ and $\overline{BE}$ are chords. A chord such as $\overline{BE}$ that passes through the center is called a ***diameter***. Since $BE = EO + OB$, a diameter is twice as long as a radius.

Key Fact J2

If *d* is the diameter and *r* is the radius of a circle: $d = 2r$.

Key Fact J3

Diameters are the longest line segments that can be drawn that have both endpoints on or inside a circle.

CIRCUMFERENCE AND AREA

The total length around a circle is called the ***circumference***. In every circle, the ratio of the circumference to the diameter is exactly the same and is denoted by the symbol **π** (the Greek letter "pi").

Key Fact J4

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{C}{d}.$$

So there there are two formulas for the circumference of a circle:

$$C = \pi d \text{ and } C = 2\pi r$$

Key Fact J5

The value of π is *approximately* 3.14.

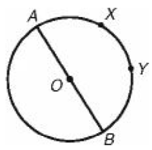
> **Smart Strategy**
>
> On almost all questions on the Math 1 test that involve circles, you are expected to leave your answer in terms of π. If you need an approximation to test the values of the choices use your calculator. To avoid rounding errors, use the π key, not 3.14.

EXAMPLE 2: If the circumference of a circle is equal to the perimeter of a square whose sides are 12, what is the radius of the circle?

Since the perimeter of the square is $4 \times 12 = 48$:

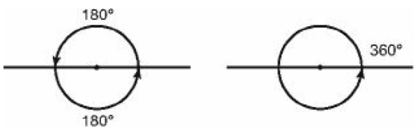
$$2\pi r = 48 \Rightarrow r = \frac{48}{2\pi} = \frac{24}{\pi} \approx 7.64$$

An ***arc*** consists of two points on a circle and all the points between them. If two points, such as *A* and *B* in circle *O*, are the endpoints of a diameter, they divide the circle into two arcs called ***semicircles***. On the Math 1 test, arc $\widehat{XY}$ always refers to the smaller arc joining *X* and *Y*. In the figure below if we wanted to refer to the larger arc going from *X* to *Y*, the one through *A* and *B*, we would refer to it as arc $\widehat{XAY}$ or arc $\widehat{XBY}$.

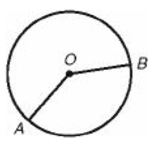


Key Fact J6

The degree measure of a circle is 360°.



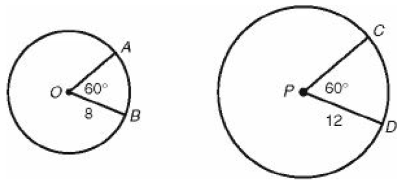
An angle such as $\angle AOB$ in the figure below, whose vertex is at the center of a circle, is called a ***central angle***.



Key Fact J7

The degree measure of an arc equals the degree measure of the central angle that intercepts it.

Degree measure is not a measure of length. In the circles at the left, arc $\widehat{AB}$ and arc $\widehat{CD}$ each measure 60° even though arc $\widehat{CD}$ is much longer.



In the figure above, how long is arc $\widehat{CD}$? Since the radius of circle *P* is 12, its diameter is 24 and its circumference is 24π. Since there are 360° in a circle, arc $\widehat{CD}$ is $\frac{60}{360}$, or $\frac{1}{6}$, of the circumference: $\frac{1}{6}(24\pi) = 4\pi$.

Key Fact J8

The formula for the area of a circle of radius *r* is *A* = π*r*2.

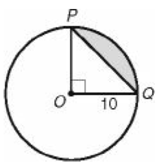
The area of circle *P*, on Key Fact J7, is $\pi(12)^2 = 144\pi$ square units. A ***sector*** is a region of a circle bounded by two radii and an arc. In that circle, since the measure of ∠*CPD* is 60° ($\frac{1}{6}$ of the circle), the area of sector *CPD* is $\frac{1}{6}$ the area of the circle: $\frac{1}{6}(144\pi) = 24\pi$.

Key Fact J9

In a circle of radius *r*, if an arc measures *x*°:

- **The length of the arc is** $\frac{x}{360}(2\pi r)$
- **The area of the sector formed by the arc and two radii is** $\frac{x}{360}(\pi r^2)$.

EXAMPLE 3: What are the perimeter and area of the shaded region in the figure below?



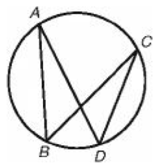
Smart Strategy

On the Math 1 test, the answer choices to questions such as these are almost always given in terms of π and square roots, so you do not need to use your calculator to evaluate them.

The circumference of the circle is $2\pi r = 2(10)\pi = 20\pi$. Since arc $\widehat{PQ}$ is $\frac{90}{360} = \frac{1}{4}$ of the circle, the length of arc $\widehat{PQ} = \frac{1}{4}(20\pi) = 5\pi$. Since PQ is the hypotenuse of isosceles right triangle *POQ*, by KEY FACT H7, $PQ = 10\sqrt{2}$. So the perimeter of the shaded region is $10\sqrt{2} + 5\pi$.

Since the area of the circle is $\pi r^2 = \pi(10^2) = 100\pi$, the area of sector *POQ* is $\frac{1}{4}(100\pi) = 25\pi$. The area of $\Delta POQ = \frac{1}{2}(10)(10) = 50$. So the area of the shaded region is $25\pi - 50$.

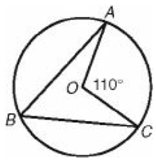
An angle formed by two chords with a common endpoint is called an ***inscribed angle***. In the figure below, ∠*ABC*, ∠*ADC*, ∠*BAD*, and ∠*BCD* are all inscribed angles.



Key Fact J10

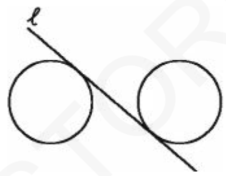
The measure of an inscribed angle is one-half the measure of its intercepted arc.

EXAMPLE 4: To find m$\angle ABC$ in circle O in the figure below, observe that since the measure of an arc is equal to the measure of the central angle that intercepts it, the measure of arc $\overset{\frown}{AC}$ is 110°. Since $\angle ABC$ is an inscribed angle, its measure is one-half the measure of arc $\overset{\frown}{AC}$: $\frac{1}{2}(110°) = 55°$.

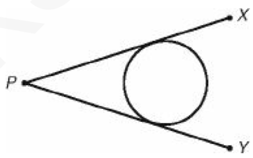


TANGENTS TO A CIRCLE

A line or line segment is ***tangent*** to a circle if it intersects the circle exactly once.



Line ℓ is tangent to both circles.



$\overline{PX}$ and $\overline{PY}$ are each tangent to the circle.

The following KEY FACT lists four important theorems about tangents.

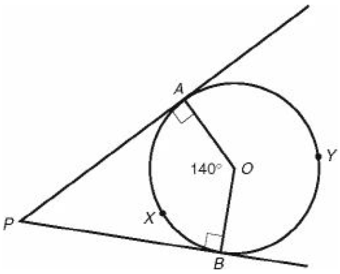
Key Fact J11

- **From any point outside a circle, exactly two tangents can be drawn to the circle.**
- **If two tangents are drawn from a point *P* outside a circle, intersecting the circle at *A* and *B*, then *PA* = *PB*.**
- **The measure of the angle formed by two tangents drawn from the same point is one-half the difference of the two intercepted arcs.**
- **A line tangent to a circle is perpendicular to the radius (or diameter) drawn to the point of contact.**

EXAMPLE 5: In the diagram below $\overline{PA}$ and $\overline{PB}$ are tangent to circle O. What is m$\angle P$?

By the fourth theorem in KEY FACT J11, m$\angle PAO$ = m$\angle PBO$ = 90°. Since the sum of the measures of the four angles in quadrilateral $PAOB$ is 360°:

$$90° + 140° + 90° + m\angle P = 360° \Rightarrow m\angle P = 40°$$

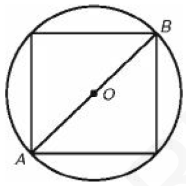


Another way to evaluate m∠ P is to observe that since ∠AOB is a central angle, the measure of arc $\widehat{AXB}$ is 140° and the measure of arc $\widehat{AYB}$ is 360° – 140° = 220°. Then, by the third theorem in KEY FACT J11:

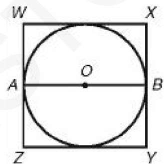
$$m\angle P = \frac{1}{2}(220° - 140°) = \frac{1}{2}(80°) = 40°$$

Key Fact J12

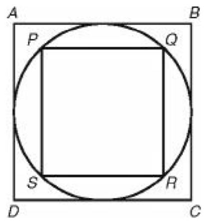
- **When a square is inscribed in a circle, the diagonals of the square are diameters of the circle. ($\overline{AB}$ is a diagonal and a diameter.)**



- **When a circle is inscribed in a square, the length of a diameter is equal to the length of a side of the square. (*AB* = *WX*)**

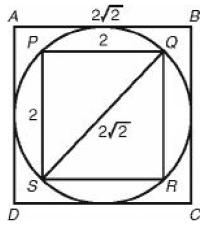


EXAMPLE 6: In the diagram below, the circle is inscribed in square *ABCD*, and square *PQRS* is inscribed in the circle.



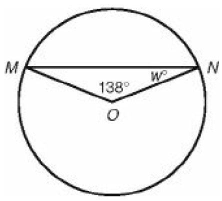
What is the ratio of the area of the large square to the area of the small square?

Choose any number for the sides of square *PQRS*, say *PQ* = 2.



Then the area of square *PQRS* is 4. Since ΔPQS is an isosceles right triangle, $QS = 2\sqrt{2}$. Since diagonal $\overline{QS}$ is a diameter of the circle, by the second statement in KEY FACT J12, $2 = \sqrt{2}$ is also the length of side $\overline{AB}$ of square *ABCD*. So the area of square *ABCD* is $(2\sqrt{2})^2 = 8$. Finally, the ratio of the area of square *ABCD* to the area of square *PQRS* is 8 : 4 or 2 : 1.

Exercises



1. In the figure above, if *O* is the center of the circle, what is the value of *w*?

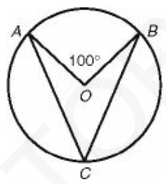
(A) 21

(B) 38

(C) 42

(D) 69

(E) 111



2. In the figure above, if *O* is the center of the circle, what is m∠*C*?

(A) 40°

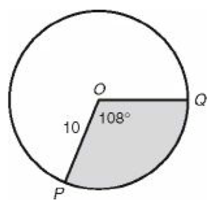
(B) 50°

(C) 80°

(D) 100°

(E) It cannot be determined from the given information.

Questions 3 and 4 refer to the following figure:



3. What is the length of arc $\widehat{PQ}$?

(A) 6

(B) 12

(C) 6π

(D) 12π

(E) 18π

4. What is the area of the shaded sector?

(A) 12

(B) 18

(C) 12π

(D) 18π

(E) 30π

5. What is the area of a circle whose circumference is $\frac{\pi}{2}$?

(A) $\frac{\pi}{16}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$

(D) π

(E) 4π

6. What is the circumference of a circle whose area is 100π?

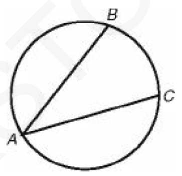
(A) 10

(B) 20

(C) 10π

(D) 20π

(E) 25π



Note: Figure not drawn to scale.

7. In the figure above, the ratio of the measure of arc $\overset{\frown}{AB}$ to the measure of arc $\overset{\frown}{BC}$ to the measure of arc $\overset{\frown}{CA}$ is 4 to 3 to 5. What is m$\angle A$?

(A) 30°

(B) 45°

(C) 60°

(D) 75°

(E) 90°

8. A square of area 2 is inscribed in a circle. What is the area of the circle?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) π

(D) $\pi\sqrt{2}$

(E) 2π

9. What is the area of a circle that is inscribed in a square of area 2?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) π

(D) $\pi\sqrt{2}$

(E) 2π

10. If *A* is the area and *C* is the circumference of a circle, which of the following is an expression for *C* in terms of *A* ?

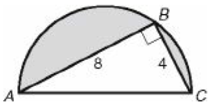
(A) $2\sqrt{\frac{\pi}{A}}$

(B) $\frac{2\pi}{\sqrt{A}}$

(C) $2\sqrt{\frac{A}{\pi}}$

(D) $2\pi\sqrt{A}$

(E) $2\sqrt{\pi A}$



11. In the figure above, triangle *ABC* is inscribed in a semicircle. What is the area of the shaded region?

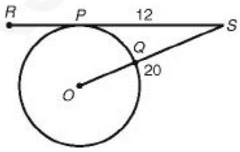
(A) 15.4

(B) 48.6

(C) 71.4

(D) 93.7

(E) 109.7



Note: Figure not drawn to scale.

12. In the figure above, $\overline{RS}$ is tangent to circle *O* at point *P*. If *OS* = 20 and *PS* = 12, what is *QS*?

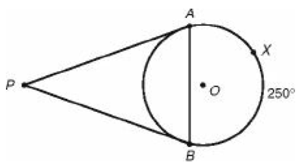
(A) 4

(B) 8

(C) 9

(D) 10

(E) 11



Note: Figure not drawn to scale

13. In the figure above, $\overline{PA}$ and $\overline{PB}$ are tangents to circle *O* and the measure of arc $\widehat{AXB}$ is 250°. What is $m\angle PAB$?

(A) 35°

(B) 55°

(C) 70°

(D) 90°

(E) 110°

14. The circumference of a circle is $x\pi$ units, and the area of the circle is $y\pi$ square units. If $y = 2x$, what is the radius of the circle?

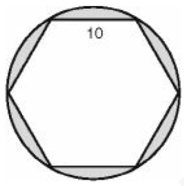
(A) 1

(B) 2

(C) 4

(D) 6

(E) 8



15. The figure above shows a regular hexagon whose sides are 10 inscribed in a circle. What is the area of the shaded region, rounded to the nearest whole number?

(A) 24

(B) 34

(C) 54

(D) 64

(E) 164

ANSWERS EXPLAINED

Answer Key

1. **(A)**	6. **(D)**	11. **(A)**
2. **(B)**	7. **(B)**	12. **(A)**
3. **(C)**	8. **(C)**	13. **(B)**
4. **(E)**	9. **(B)**	14. **(C)**
5. **(A)**	10. **(E)**	15. **(C)**

Solutions

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the "Tactics" chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(A)** $m\angle M = m\angle N + 138 = 180 \Rightarrow m\angle M = m\angle N = 42$. Since $\overline{OM}$ and $m\angle M = m\angle N = w$ are radii, $OM = ON$, ΔOMN is isosceles, and $m\angle M = m\angle N = w$. Therefore, $2w = 42 \Rightarrow w = 21$.

2. **(B)** Since $\angle AOB$ is a central angle, m$\widehat{AB}$ = 100°. Since $\angle ACB$ is an inscribed angle, $m\angle C = \frac{1}{2}(100°) = 50°$.

3. **(C)** The length of arc $\widehat{PQ}$ is $\frac{108}{360}=\frac{3}{10}$ of the circumference, which is 2π(10) = 20π. So the length of arc $\widehat{PQ}=\frac{3}{10}(20\pi)=6\pi$.

4. **(E)** The area of the shaded sector is $\frac{108}{360} = \frac{3}{10}$ of the area of the circle, which is $\pi(10)^2 = 100\pi$. So the area of the shaded sector is $\frac{3}{10}(100\pi) = 30\pi$.

5. **(A)** $C = 2\pi r = \frac{\pi}{2} \Rightarrow 2r = \frac{1}{2} \Rightarrow r = \frac{1}{4} \Rightarrow A = \pi r^2 = \pi\left(\frac{1}{4}\right)^2 = \frac{1}{16}\pi = \frac{\pi}{16}$

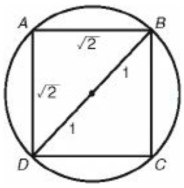
6. **(D)** $A = \pi r^2 = 100\pi \Rightarrow r^2 = 100 \Rightarrow r = 10 \Rightarrow C = 2\pi r = 2\pi(10) = 20\pi$

7. **(B)** The sum of the measures of the three arcs is 360°. So:

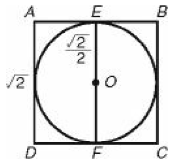
$$4x + 3x + 5x = 360° \Rightarrow 12x = 360° \Rightarrow x = 30°$$

So the measure of arc $\widehat{BC}$ is $3 \times 30° = 90°$, and $m\angle A = \frac{1}{2}(90°) = 45°$.

8. **(C)** Draw a picture. Since the area of square *ABCD* is 2, *AD* = $\sqrt{2}$. So diagonal *BD* = $\sqrt{2} \times \sqrt{2}$ = 2. $\overline{BD}$ is also a diameter of the circle. So the diameter is 2 and the radius is 1. Therefore, the area is $\pi(1)^2 = \pi$.



9. **(B)** Draw a picture. Since the area of square *ABCD* is 2, $AD = \sqrt{2}$.



Then diameter $EF = \sqrt{2} \Rightarrow$ radius $OE = \frac{\sqrt{2}}{2} \Rightarrow$ the area of circle *O* is

$$\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4}\pi = \frac{\pi}{2}.$$

10. **(E)** $A = \pi r^2 \Rightarrow r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}} = \frac{\sqrt{A}}{\sqrt{\pi}}$

$$C = 2\pi r = 2\pi \frac{\sqrt{A}}{\sqrt{\pi}} = \frac{2\pi\sqrt{A}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} = \frac{2\cancel{\pi}\sqrt{A\pi}}{\cancel{\pi}} = 2\sqrt{\pi}\sqrt{A} = 2\sqrt{\pi A}$$

11. **(A)** Since an angle inscribed in a semicircle is a right angle, ΔABC is a right triangle. By the Pythagorean theorem:

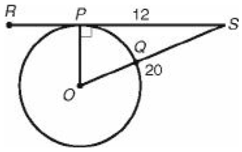
$$(AC)^2 = 4^2 + 8^2 = 16 + 64 = 80 \Rightarrow AC = \sqrt{80}$$

Since $\overline{AC}$ is a diameter, the radius is $\frac{\sqrt{80}}{2}$ and the area of the semicircle is

$\frac{1}{2}\pi\left(\frac{\sqrt{80}}{2}\right)^2 = \frac{1}{2}\pi\frac{80}{4} = 10\pi$. The area of right triangle *ABC* is $\frac{1}{2}(8)(4) = 16$.

So the area of the shaded region is $10\pi - 16 \approx 31.4 - 16 = 15.4$.

12. **(A)** Since $\overline{OP} \perp \overline{RS}$, ΔOPS is a right triangle. By the Pythagorean theorem:
$OP^2 + PS^2 = OS^2 \Rightarrow OP^2 + 144 = 400$ and so $OP^2 = 256 \Rightarrow OP = 16$



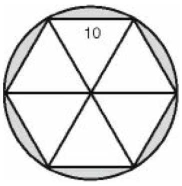
Since all radii are equal, $OQ = 16 \Rightarrow QS = 20 - 16 = 4$.

13. **(B)** The measure of the smaller arc from *A* to *B* is 360° – 250° = 110°. So m$\angle P = \frac{1}{2}(250° - 110°) = \frac{1}{2}(140°) = 70°$. Since the two tangents are congruent, Δ*PAB* is isosceles and the measure of each base angle is $\frac{1}{2}(180° - 70°) = \frac{1}{2}(110°) = 55°$.

14. **(C)** Since $C = x\pi$, we have $x\pi = 2\pi r \Rightarrow x = 2r$. Since $A = y\pi$, we have $y\pi = \pi r^2 \Rightarrow y = r^2$. Finally, since it is given that $y = 2x$,

$$y = 2x \Rightarrow r^2 = 2(2r) = 4r \Rightarrow r = 4.$$

15. **(C)** Three diameters divide the hexagon into 6 equilateral triangles.



By using the formula for the area of an equilateral triangle, we find that the area of the hexagon is $6\left(\frac{10^2\sqrt{3}}{4}\right)=6\left(\frac{100\sqrt{3}}{4}\right)=150\sqrt{3}$. Since the radius of the circle is 10, its area is $\pi(10^2) = 100\pi$. So the area of the shaded region is $100\pi - 150\sqrt{3} \approx 314 - 260 = 54$.

SOLID AND COORDINATE GEOMETRY

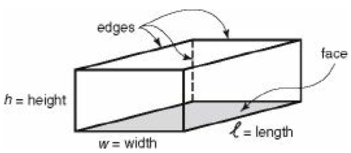
Solid Geometry

On a typical Math 1 test, there are about three questions on solid geometry. Most of them can be answered if you know the formulas for the volumes and surface areas of rectangular solids and cylinders. Although these formulas are given to you on the EST they are *not* provided on the Math 1 test, so you must learn them.

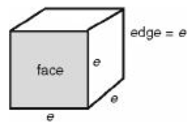
Occasionally, a Math 1 test has one question about spheres, cones, or pyramids. The formulas you would need to answer such a question *are* given to you in a box labeled “Reference Information” at the beginning of the test. However, many Math 1 tests have no questions at all on these topics, so if your study time is limited, you may want to skip the discussion of these solids.

RECTANGULAR SOLIDS

A ***rectangular solid*** or ***box*** is a solid formed by six rectangles, called ***faces***. The sides of the rectangles are called ***edges***. As in the diagram below, the edges are called the ***length***, ***width***, and ***height***. A ***cube*** is a rectangular solid in which the length, width, and height are equal, so all the edges are the same length.



RECTANGULAR SOLID



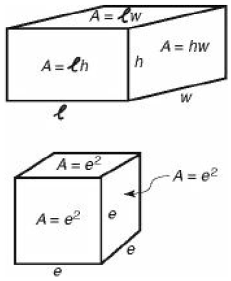
CUBE

The ***volume*** of a solid is measured in ***cubic units***. One cubic unit is the amount of space taken up by a cube all of whose edges are one unit long. In the figure on RECTANGULAR SOLIDS, if each edge of the cube is 1 inch, the area of each face is 1 square inch, and the volume of the cube is 1 cubic inch.

Key Fact K1

- **The formula for the volume of a rectangular solid is $V = \ell wh$, where ℓ, w, and h are the length, width, and height, respectively.**
- **Since all the edges of a cube are equal, if e is an edge, the formula for the volume of a cube is $V = e^3$.**

The ***surface area*** of a rectangular solid is the sum of the areas of the six rectangular faces. The areas of the top and bottom faces are equal, the areas of the front and The back faces are equal, and the areas of the left and right faces are equal. Therefore, to get the total surface area, we can calculate the area of one face from each pair, add them up, and then double the sum. In a cube, each of the six faces has the same area, so the surface area is six times the area of any face.



Key Fact K2

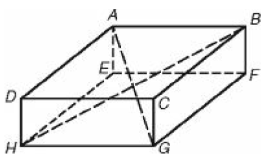
- **The formula for the surface area of a rectangular solid is $A = 2(\ell w + \ell h + wh)$.**
- **The formula for the surface area of a cube is $A = 6e^2$.**

EXAMPLE 1: Assume that the surface area of a cube is x square inches, that the volume of that cube is y cubic inches, and that $y = 2x$. If the length of an edge of the cube is e inches, we have $x = 6e^2$ and $y = e^3$. Then

$$y = 2x \Rightarrow e^3 = 2(6e^2) = 12e^2$$

Dividing both sides of this equation by e^2, we get $e = 12$. So each edge is 12 inches or 1 foot long.

A ***diagonal*** of a rectangular solid is a line segment joining a vertex on one face of the box to the vertex on the opposite face that is furthest away. A rectangular solid has four diagonals, all the same length. In the following box, diagonals $\overline{AG}$ and $\overline{BH}$ are drawn in. The other two diagonals are $\overline{CE}$ and $\overline{DF}$.

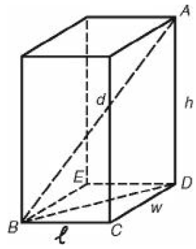


Key Fact K3

If the dimensions of a rectangular solid are ℓ, w, and h and if d is the length of a diagonal, then $d = \sqrt{\ell^2 + w^2 + h^2}$.

The formula given in KEY FACT K3 is obtained by using the Pythagorean theorem twice. In the figure below, $\overline{BD}$ is the diagonal of rectangular face $BCDE$. By $\overline{BD}$ the Pythagorean theorem, $(BD)^2 = \ell^2 + w^2$. Now ΔADB is a right triangle, and by the Pythagorean theorem:

$$d^2 = (AB)^2 = (BD)^2 + (AD)^2 = (\ell^2 + w^2) + h^2 \Rightarrow d = \sqrt{\ell^2 + w^2 + h^2}$$



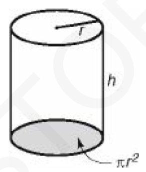
TIP

The distance between *any* two points can *always* be calculated using the Pythagorean theorem.

EXAMPLE 2: To determine the length of a diagonal of a cube whose sides are *e*, use the formula from KEY FACT K3, noting that $\ell = w = h = e$. Then, $d = \sqrt{e^2 + e^2 + e^2} = \sqrt{3e^2} = e\sqrt{3}$.

CYLINDERS

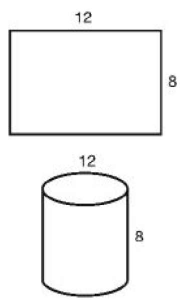
A ***cylinder*** is similar to a rectangular solid except that the base is a circle instead of a rectangle. To find the volume of a rectangular solid, we multiply the area of its rectangular base, ℓw, by its height, *h*. For a cylinder, we do exactly the same thing. The volume of a cylinder is the area of its circular base, πr^2, times its height, *h*. The surface area of a cylinder depends on whether you are envisioning a tube, such as a straw without a top and bottom, or a can, which has both a top and bottom.



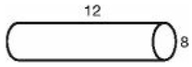
Key Fact K4

- **The formula for the volume, *V*, of a cylinder whose circular base has radius *r* and whose height is *h* is $V = \pi r^2 h$.**
- **The formula for the surface area, *A*, of the side of a cylinder is the product of the circumference of its circular base and its height: $A = 2\pi rh$.**
- **The areas of the top and bottom of a cylinder are each πr^2, so the total surface area of a cylindrical can is $2\pi rh + 2\pi r^2$.**

EXAMPLE 3: You can roll an 8 x 12 rectangular piece of paper into a cylinder in two ways. You could tape the 8-inch sides together, or you could tape the 12-inch sides together. Note that these cylinders do *not* have the same volume.



Cylinder I



Cylinder II

In cylinder I, $C = 12 \Rightarrow 2\pi r = 12 \Rightarrow r = \frac{6}{\pi}$, and so

$$V = \pi\left(\frac{6}{\pi}\right)^2(8) = \pi\left(\frac{36}{\pi^2}\right)8 = \frac{288}{\pi}.$$

In cylinder II, $C = 8 \Rightarrow 2\pi\, r = 8 \Rightarrow r = \frac{4}{\pi}$, and so

$$V = \pi\left(\frac{4}{\pi}\right)^2(12) = \pi\left(\frac{16}{\pi^2}\right)12 = \frac{192}{\pi}.$$

PRISMS

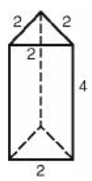
Rectangular solids and cylinders are special cases of geometric solids called prisms. A ***prism*** is a three-dimensional figure that has two congruent parallel ***bases***. The (perpendicular) distance between the two bases is called the ***height***. Four prisms are depicted in the figure below.

On the Math 1 test, all of the prisms are ***right prisms***, which means that any line segment joining corresponding points on the bases is perpendicular to the bases. In the figure above, the first three prisms are right prisms; the fourth one is not. The volume formulas given in KEY FACTS K1 and K4 are special cases of the following formula.

Key Fact K5

The formula for the volume of any right prism is $V = Bh$, where B is the area of one of the bases and h is the height.

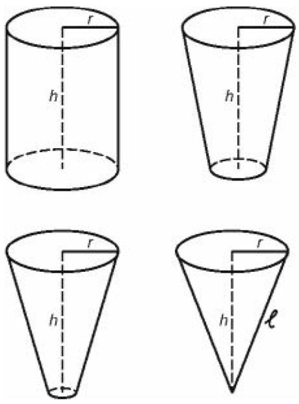
EXAMPLE 4: What is the volume of the triangular prism below?



First calculate the area of a base and multiply it by the height. By KEY FACT H11, the area of an equilateral triangle whose sides are 2 is $\frac{2^2\sqrt{3}}{4} = \sqrt{3}$. So the volume is $(\sqrt{3})(4) = 4\sqrt{3}$.

CONES

Imagine taking a cylinder and shrinking the size of one of the circular bases.



When it shrinks to a point, we call the resulting solid a cone. If you picture an ice cream cone (without the ice cream) or a dunce cap, you have the right idea. Notice that the radius of the circular base and the height of the cone are the same as in the original cylinder. In the figure above, ℓ is called the slant height of the cone.

In the Math 1 "Reference Information," cones are referred to as "right circular cones." This is done to emphasize that the base is a circle and that the height (the line segment joining the vertex and the center of the circular base) is perpendicular to the base. KEY FACT K6 lists the two formulas concerning cones that are given to you in the "Reference Information" box on the first page of the test and gives an alternative for one of them. Note that in the second formula, the circumference of the circular base is denoted by a lowercase "*c*," whereas we usually use an uppercase "*C*" to represent circumference. Also, ℓ, the ***slant height***, is the distance from any point on the circumference to the vertex of the cone.

Key Fact K6

- **The formula for the volume of a right circular cone is** $V = \frac{1}{3}\pi r^2 h$.
- **The formula for the lateral surface area of a right circular cone is ,where *c* is the circumference and** ℓ **is the slant height.**
- **Since** $c = 2\pi r$; $\frac{1}{2}c = \pi r$**, so an alternative formula for the lateral surface area is** $A = \pi r\ell$.
- **The total surface area of a cone is the sum of its lateral area and the area of its circular base:** $SA = \pi r\ell + \pi r^2$.

The first two formulas in KEY FACT K6 are listed in the Reference Information box on the first page of every Math 1 test.

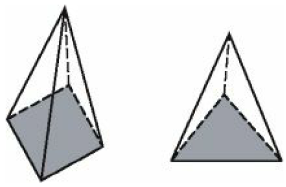
EXAMPLE 5: Assume that the volumes of a right circular cone and a right circular cylinder are equal and that the radius of the cone is twice the radius of the cylinder. How do their heights compare?

Let *r* be the radius of the cylinder, 2*r* the radius of the cone, and *h* and *H* the heights of the cone and the cylinder, respectively. Then

$$\frac{1}{3}\pi(2r)^2 h = \pi r^2 H \Rightarrow \frac{1}{3}\not{\pi}4\not{r^2}h = \not{\pi}\not{r^2}H \Rightarrow \frac{4}{3}h = H$$

PYRAMIDS

A ***pyramid*** is very similar to a cone. The difference is that the base of a pyramid is a polygon, not a circle. If there is a question concerning pyramids on the Math 1 test you take, the polygonal base will almost surely be a square or a triangle.



The formula for the volume of a pyramid, which is one of the five formulas in the "Reference Information" box on the Math 1 test, is given in KEY FACT K7.

Key Fact K7

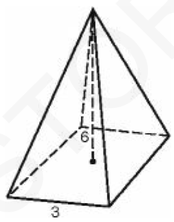
The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where *B* is the area of the base and *h* is the height.

The formula in KEY FACT K7 is listed in the Reference Information box on the first page of every Math 1 test.

Note that since the area of the circular base of a cone is r^2, this formula applies to cones, as well.

EXAMPLE 6: What is the volume of a pyramid whose base is a square of side 3 feet and whose height is 6 feet?

Sketch the pyramid and use the formula given in KEY FACT K7.

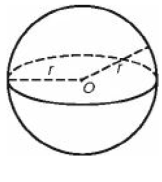


The area of the square base is 9 square feet, and so $V = \frac{1}{3}(9)(6) = 18$ cubic feet.

SPHERES

A sphere is the set of all points in space that are a fixed distance, *r*, from a given point, *O*. *O* is called the center of the sphere, and *r* is the radius.

The two formulas about spheres that you might need to use on the Math 1 test are those for volume and surface area. These two formulas, which are given to you in the "Reference Information" box, constitute the next KEY FACT.



Key Fact K8

- **The formula for the volume of a sphere of radius *r* is** $V = \frac{4}{3}\pi r^3$.
- **The formula for the surface area of a sphere of radius *r* is $A = 4\pi r^2$.**

The formulas in KEY FACT K8 are listed in the Reference Information box on the first page of every Math 1 test.

EXAMPLE 7: What is the radius of a sphere whose surface area is equal to the surface area of a cube whose edges are 1?

By KEY FACT K2, the surface area of the cube is $6e^2 = 6(1)^2 = 6$. Then by KEY FACT K8,

$$4\pi r^2 = 6 \Rightarrow r^2 = \frac{6}{4\pi} \Rightarrow r = \sqrt{\frac{6}{4\pi}} \approx 0.69$$

Exercises

1. What is the volume of a cube whose surface area is 54?

(A) 27

(B) 36

(C) 54

(D) 64

(E) 216

2. What is the surface area of a cube whose volume is 216?

(A) 6

(B) 36

(C) 72

(D) 144

(E) 216

3. What is the volume of a sphere whose surface area is 36π?

(A) $\frac{4}{3}\pi$

(B) 9π

(C) 36π

(D) 72π

(E) 108π

4. What is the surface area of a sphere whose volume is 288π?

(A) 6π

(B) 36π

(C) 72π

(D) 144π

(E) 288π

5. What is the lateral surface area of a right circular cone whose radius is 3 and whose height is 4?

(A) 12π

(B) 15π

(C) 24π

(D) 30π

(E) 36π

6. The height, *h*, of a right circular cylinder is equal to the edge of a cube. If the cylinder and cube have the same volume, what is the radius of the cylinder?

(A) $\frac{h}{\sqrt{\pi}}$

(B) $h\sqrt{\pi}$

(C) $\frac{\sqrt{\pi}}{h}$

(D) $\frac{h^2}{\pi}$

(E) πh^2

7. What is the volume of a pyramid whose base is a square of area 36 and whose four faces are equilateral triangles?

(A) $24\sqrt{2}$

(B) 36

(C) $36\sqrt{2}$

(D) 108

(E) $108\sqrt{2}$

8. An isosceles right triangle whose legs are 6 is rotated about one of its legs to generate a right circular cone. What is the volume of that cone?

(A) $48\sqrt{2}\pi$

(B) 72π

(C) $72\sqrt{2}\pi$

(D) 144π

(E) 216π

9. If the height of a right circular cylinder is 4 times its circumference, what is the volume of the cylinder in terms of its circumference, C ?

(A) $\frac{C^3}{\pi}$

(B) $\frac{2C^3}{\pi}$

(C) $\frac{2C^2}{\pi^2}$

(D) $\frac{\pi C^2}{4}$

(E) $4\pi C^3$

10. Three identical balls fit snugly into a cylindrical can: the radius of the spheres equals the radius of the can, and the balls just touch the bottom and top of the can. What fraction of the volume of the can is taken up by the balls?

(A) $\frac{1}{3}$

(B) $\frac{4}{9}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

(E) $\frac{3}{4}$

ANSWERS EXPLAINED

Answer Key

1. **(A)**	5. **(B)**	9. **(A)**
2. **(E)**	6. **(A)**	10. **(D)**
3. **(C)**	7. **(C)**	
4. **(D)**	8. **(B)**	

Solutions

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the "Tactics" chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(A)** Since the surface area is 54, each of the 6 faces is a square whose area is 54 ÷ 6 = 9. So the edges are all 3, and by KEY FACT K1, the volume is $3^3 = 27$.

2. **(E)** Since the volume of the cube is 216, we have $e^3 = 216 \Rightarrow e = 6$. By KEY FACT K2, the surface area is $6e^2 = 6 \times 6^2 = 6 \times 36 = 216$.

3. **(C)** Since the surface area of the sphere is 36π, by KEY FACT K8:

$$4\pi r^2 = 36\pi \Rightarrow r^2 = 9 \Rightarrow r = 3.$$

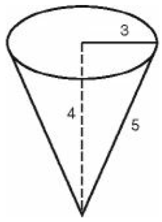
$$\text{So } V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(27) = 36\pi$$

4. **(D)** Since the volume of the sphere is 288π, by KEY FACT K8, we have:

$$\frac{4}{3}\pi r^3 = 288\pi \Rightarrow r^3 = \frac{3}{4}(288) = 216 \Rightarrow r = \sqrt[3]{216} = 6.$$

So $A = 4\pi r^2 = 4\pi(36) = 144\pi$

5. **(B)** First sketch the cone.

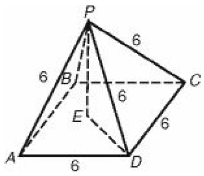


By the Pythagorean theorem, the slant height is 5. The circumference of the circular base is $2\pi(3) = 6\pi$. So by KEY FACT K6, the lateral surface area is: $\frac{1}{2}\ell d = \frac{1}{2}(6\pi)5 = 15\pi$.

6. **(A)** Since the volumes are equal, $\pi r^2 h = e^3 = h^3$. So:

$$\pi r^2 = h^2 \Rightarrow r^2 = \frac{h^2}{\pi} \Rightarrow r = \frac{h}{\sqrt{\pi}}$$

7. **(C)** Sketch the pyramid.



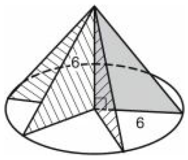
Since the area of the square is 36, each side is 6; and since each triangular face is equilateral, each edge of the pyramid is 6. To find the height of the pyramid, use the Pythagorean theorem on $\triangle PED$. Since $\overline{ED}$ is half of diagonal $\overline{BD}$, $ED = \frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}$. So:

$$(PD)^2 = (PE)^2 + (ED)^2 \Rightarrow (6)^2 = h^2 + (3\sqrt{2})^2 \Rightarrow$$
$$36 = h^2 + 18 \Rightarrow h^2 = 18 \Rightarrow h = \sqrt{18} = 3\sqrt{2}$$

Finally, by KEY FACT K7, the volume of the pyramid is:

$$\frac{1}{3}Bh = \frac{1}{\cancel{3}}(36)(\cancel{3}\sqrt{2}) = 36\sqrt{2}$$

8. **(B)** Sketch the triangle and cone.



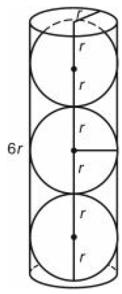
Since the radius and height of the cone are 6, by KEY FACT K6:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6)^2(6) = 72\pi$$

9. **(A)** Since $V = \pi r^2 h$, we need to express r and h in terms of C. It is given that $h = 4C$, and since $C = 2\pi r$, $r = \frac{C}{2\pi}$. So:

$$V = \pi\left(\frac{C}{2\pi}\right)^2(4C) = \pi\left(\frac{C^2}{4\pi^2}\right)(4C) = \frac{C^3}{\pi}$$

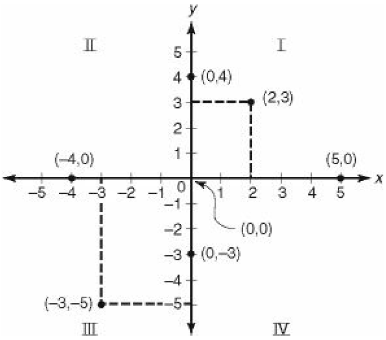
10. **(D)** The volume of each ball is $\frac{4}{3}\pi r^3$, and the total volume of the 3 balls is . The height of the can is equal to 3 times the diameter, and hence 6 times the radius, of a ball. So the volume of the can is $\pi r^2(6r) = 6\pi r^3$. Therefore, the balls take up $\frac{4\pi r^3}{6\pi r^3} = \frac{2}{3}$ of the can.



Coordinate Geometry

Of the 50 questions on the Math 1 test, approximately six are on the coordinate geometry topics specifically discussed in this chapter. In addition, three or four more questions that the College Board would classify under other topics, such as functions and their graphs or even probability or trigonometry, could be considered as coordinate geometry questions. So this is an important topic that you should review thoroughly.

The coordinate plane is formed by two perpendicular number lines called the ***x-axis*** and ***y-axis***, which intersect at the ***origin***. The axes divide the plane into four ***quadrants***, labeled I, II, III, and IV as shown in the figure below. Each point in the plane is assigned two numbers, an ***x-coordinate*** and a ***y-coordinate***, which are written as an ordered pair, **(*x, y*)**. The point (2, 3) has an *x*-coordinate of 2 and a *y*-coordinate of 3 and is located at the intersection of the vertical line that crosses the *x*-axis at 2 and the horizontal line that crosses the *y*-axis at 3.



Key Fact L1

- **The *y*-coordinate of every point on the *x*-axis is 0.**
- **The *x*-coordinate of every point on the *y*-axis is 0.**

• **To find the *x*-intercepts of a graph, replace *y* by 0 in the equation of the graph.**

• **To find the *y*-intercepts of a graph, replace *x* by 0 in the equation of the graph.**

EXAMPLE 1: Where does the graph of $y = x^2 - 2x - 3$ cross the *x*-axis and where does it cross the *y*-axis?

By KEY FACT L1, the *x*-intercepts are the points whose *y*-coordinates are 0. So replace *y* with 0:

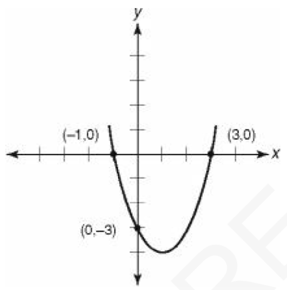
The *x*-intercepts of this graph are the points (–1, 0) and (3, 0).

By KEY FACT L1, the *y*-intercept is the point whose *x*-coordinate is 0. So replace *x* with 0:

$$y = 0^2 - 2(0) - 3 = -3 \Rightarrow y = -3$$

The *y*-intercept is the point (0, –3).

The graph of $y = x^2 - 2x - 3$ is the parabola in the figure below, on which the *x*- and *y*-intercepts are indicated.



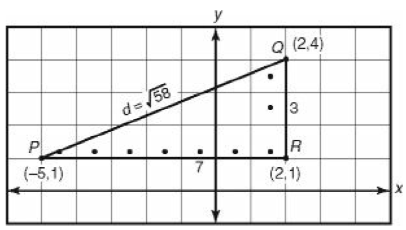
DISTANCE BETWEEN TWO POINTS

Often a question on the Math 1 test requires you to find the distance between two points. This is easiest when the points lie on the same horizontal or vertical line.

Key Fact L2

• **All the points on a horizontal line have the same *y*-coordinate. To find the distance between them, subtract their *x*-coordinates.**

• **All the points on a vertical line have the same *x*-coordinate. To find the distance between them, subtract their *y*-coordinates.**



In the diagram above, the distance from *P* to *R* is $2 - (-5) = 7$ and the distance from *R* to *Q* is $4 - 1 = 3$. If you want to find the distance between two points, such as *P* and *Q*, that do not lie on the same horizontal or vertical line, sketch the line segment joining those points and then create a right triangle with that segment as the hypotenuse by drawing a horizontal line through one of the points and a vertical line through the other. Use KEY FACT L2 to find the length of the legs. To calculate the hypotenuse, use the Pythagorean theorem. For example, in the figure above, if *d* represents the distance from *P* to *Q*,

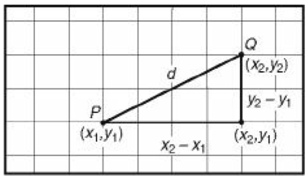
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

An alternative method is to use the distance formula given in KEY FACT L3.

Key Fact L3

The distance, *d*, between two points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, can be calculated using the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

From the diagram below, you should see that the distance formula is really just the Pythagorean theorem.



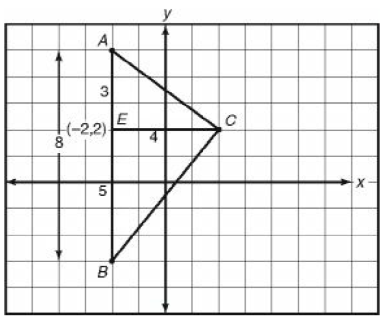
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2: If *A* (–2, 5), *B* (–2, –3), and *C* (2, 2) are the vertices of △*ABC*, what are the area and perimeter of the triangle?

Start by making a sketch, and then calculate the lengths of the three line segments.

Since *A* and *B* lie on the same vertical line, we can find *AB* by subtracting their *y*-coordinates: *AB* = 5 – (–3) = 8.

Now draw in altitude $\overline{CE}$.



Since $\overline{CE}$ is horizontal, *CE* = 2 – (–2) = 4. If we treat $\overline{AB}$ as the base of △*ABC*, then $\overline{CE}$ is the height, and the area of the triangle is $\frac{1}{2}(8)(4) = 16$.

On Example 2, we found that *AB* = 8. To find the perimeter of △*ABC*, we need the lengths of the other two sides. Each of these can be calculated using the distance formula or the Pythagorean theorem:

$$(AC)^2 = 3^2 + 4^2 = 9 + 16 = 25 \Rightarrow AC = 5$$
$$(BC)^2 = 5^2 + 4^2 = 25 + 16 = 41 \Rightarrow BC = \sqrt{41}$$

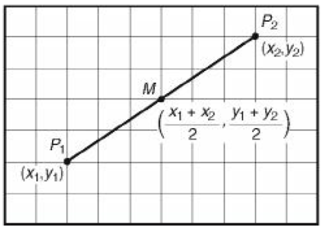
So the perimeter of $\triangle ABC$ is $8+5+\sqrt{41}=13+\sqrt{41}$

THE MIDPOINT OF A SEGMENT

Recall that the midpoint, *M*, of line segment $\overline{PQ}$ is the point on $\overline{PQ}$ such that *PM* = *MQ*. In coordinate geometry, the *x*-coordinate of the midpoint is the average of the *x*-coordinates of the two endpoints, and the *y*-coordinate of the midpoint is the average of the *y*-coordinates of the two endpoints.

Key Fact L4

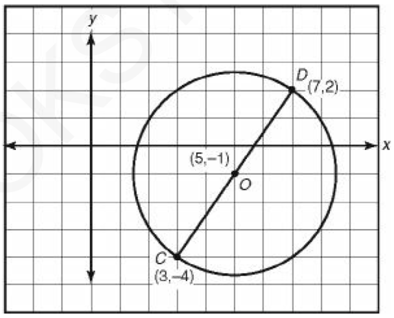
If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points, then the midpoint, *M*, of segment $\overline{P_1P_2}$ is the point whose coordinates are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.



EXAMPLE 3: If *C*(3, –4) and *D*(7,2) are the endpoints of diameter *CD* of circle *O*, what are the coordinates of *O*?

First, sketch the circle (see diagram). Since the center of a circle is the midpoint of any diameter, we can use the midpoint formula to find the coordinates of *O*.

$$\left(\frac{3+7}{2}, \frac{-4+2}{2}\right)=\left(\frac{10}{2}, \frac{-2}{2}\right)=(5,-1)$$



EXAMPLE 4: To find the area of circle *O* in Example 3, we have to use the formula $A = \pi r^2$, which means we have to determine *r*. To do this, we can either find the length of diameter $\overline{CD}$ and divide it by 2 or find the length of radius $\overline{OD}$.

$$CD=\sqrt{(7-3)^2+(2-(-4))^2}=\sqrt{4^2+6^2}=\sqrt{16+36}$$
$$=\sqrt{52} \Rightarrow r=\frac{\sqrt{52}}{2}, \text{ or}$$
$$OD=\sqrt{(7-5)^2+(2-(-1))^2}=\sqrt{2^2+3^2}=\sqrt{4+9} \Rightarrow \sqrt{13}$$

Then, depending on which expression you found,

SLOPE

The ***slope*** of a line is a number that represents how steep it is: the larger the absolute value of the slope, the steeper the line. This intuitive definition is made more precise in KEY FACT L5.

Key Fact L5

- **Vertical lines *do not have slopes*.**
- **To find the slope of any other line, proceed as follows:**

  (1) Choose any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line.

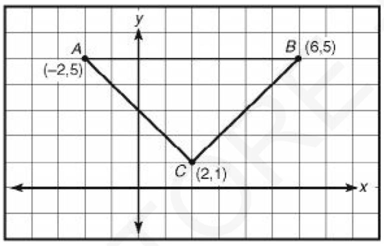
  (2) Determine the differences of their *y*–coordinates, $y_2 - y_1$, and their *x*–coordinates, $x_2 - x_1$.

  (3) Divide: slope = $\frac{y_2 - y_1}{x_2 - x_1}$.

Since $y_2 - y_1$ is the difference between the *y*–coordinates of the two points and $x_2 - x_1$ is the difference between the *x*–coordinates of the two points, we often say that the slope is “the change in *y*” over “the change in *x*”:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}.$$

We will illustrate the next KEY FACT by using the slope formula to calculate the slopes of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ in the next figure.



Key Fact L6

- **The slope of any horizontal line is 0:**

$$\text{slope of } \overline{AB} = \frac{5-5}{6-(-2)} = \frac{0}{8} = 0$$

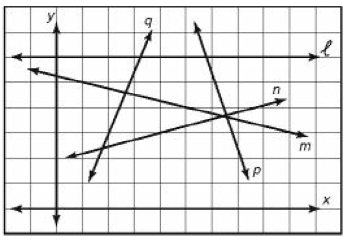
- **The slope of any line that goes up as you move from left to right is positive:**

$$\text{slope of } \overline{CB} = \frac{5-1}{6-2} = \frac{4}{4} = 1$$

- **The slope of any line that goes down as you move from left to right is negative:**

$$\text{slope of } \overline{AC} = \frac{1-5}{2-(-2)} = \frac{-4}{4} = -1$$

On the Math 1 test, you may be asked to compare the slopes of lines without being given any coordinates.



EXAMPLE 5: To list the lines shown in the previous figure in order of increasing slope, first use KEY FACT L6. Since line ℓ is horizontal, its slope is 0. Since as you move from left to right lines *m* and *p*

go down and lines *n* and *q* go up, the slopes of *m* and *p* are negative, whereas the slopes of *n* and *q* are positive. Since *q* is steeper than *n*, the slope of *q* is greater than the slope of *n*. Be careful as you compare the slopes of *m* and *p*. Line *p* is steeper than line *m*, so the absolute value of the slope of *p* is greater than the absolute value of the slope of *m*, but this means that the slope of *p* is less than the slope of *m*. (For example, |–5| > |–1|, but –5 < –1.) So listed in order of increasing slope we have:

$$\underbrace{p < m}_{\text{negative}} < \underset{0}{\ell} < \underbrace{n < q}_{\text{positive}}$$

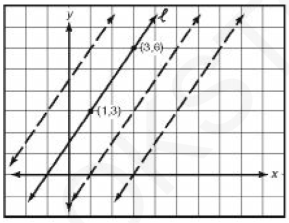
Often, a Math 1 test has at least one question concerning the slopes of parallel and/or perpendicular lines. You will have no trouble answering such questions if you know the next KEY FACT.

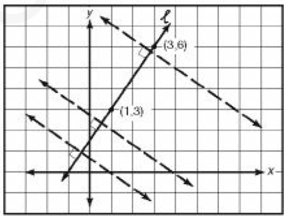
Key Fact L7

- **If two nonvertical lines are parallel, their slopes are equal.**
- **If two nonvertical lines are perpendicular, the product of their slopes is –1.**

If the product of two numbers, *a* and *b*, is –1, then $ab = -1 \Rightarrow a = -\frac{1}{b}$. So another way to express the second part of KEY FACT L7 is to say that **if two nonvertical lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other**.

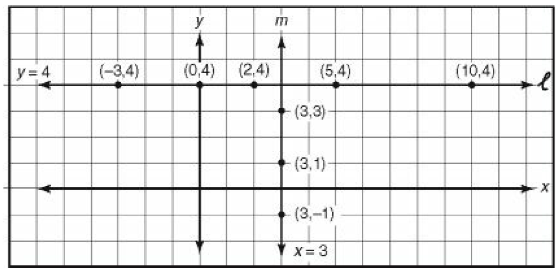
EXAMPLE 6: Let ℓ be the line that passes through points (1, 3) and (3, 6). Then the slope of ℓ is $\frac{6-3}{3-1} = \frac{3}{2}$. If $m \parallel \ell$, then the slope of *m* is also $\frac{3}{2}$. In the figure below left Key Fact L7, each of the dotted lines has a slope of $\frac{3}{2}$. If $n \perp \ell$, then the slope of *n* is $-\frac{2}{3}$, the negative reciprocal of $\frac{3}{2}$. In the figure below right Key Fact L7, each of the dotted lines has a slope of $-\frac{2}{3}$.





EQUATIONS OF LINES

Every line that is drawn in a coordinate plane has an equation. A point (*x*, *y*) is on a line if and only if the values of *x* and *y* satisfy that equation. All the points on a horizontal line have the same *y*-coordinate. For example, in the figure below, horizontal line ℓ passes through (–3, 4), (0, 4), (2, 4), (5, 4), and (10, 4). The equation of line ℓ is $y = 4$. Similarly, every point on a vertical line has the same *x*-coordinate. Each point on line *m* has an *x*-coordinate of 3. The equation of *m* is $x = 3$.



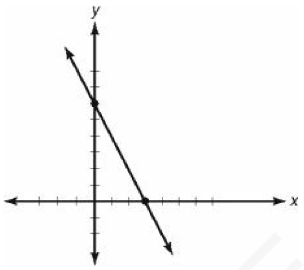
Every other line in the coordinate plane has an equation that can be written in the form $y = mx + b$, where *m* is the slope of the line and *b* is the ***y*-intercept**—the *y*-coordinate of the point where the line crosses the *y*-axis. These facts are summarized in KEY FACT L8.

Key Fact L8

- **For any real number *a*, $x = a$ is the equation of the vertical line that crosses the *x*-axis at $(a, 0)$.**
- **For any real number *b*, $y = b$ is the equation of the horizontal line that crosses the *y*-axis at $(0, b)$.**
- **For any real numbers *b* and *m*, $y = mx + b$ is the equation of the line that crosses the *y*-axis at $(0, b)$ and whose slope is *m*.**

On the Math 1 test, you could be given an equation and asked which of five graphs is the graph of that equation. Conversely, you could be given a graph and asked which of five equations is the equation of that graph.

EXAMPLE 7: Which of the following is the equation of the line in the figure below?



(A) $y = -2x + 6$
(B) $y = -\frac{1}{2}x + 6$
(C) $y = 2x + 6$
(D) $y = -2x - 6$
(E) $y = 2x - 6$

On the Math 1 test, there are two ways to answer the question:

Solution 1. Since the line is neither horizontal nor vertical, its equation has the form $y = mx + b$. Since the line crosses the *y*-axis at 6, $b = 6$. Since the line passes through (0, 6) and (3, 0), its slope, *m*, is . The equation is $y = -2x + 6$, choice A.

Solution 2. The second way to answer this question takes advantage of the fact that the Math 1 test is a multiple-choice test. Since the line passes through (0, 6), $y = 6$ when $x = 0$. Plug in 0 for *x* in the five choices, and eliminate choices D and E, the two choices for which $y \neq 6$. Then note that the line also passes through (3, 0), so when $x = 3$, $y = 0$. Replace *x* by 3 in the remaining choices (A, B, and C) and see which one equals 0. Only choice A works.

EXAMPLE 8: What is the equation of the line that passes through (1, –3) and (5, 5)?

First find the slope of the line:

$$m = \frac{5-(-3)}{5-1} = \frac{8}{4} = 2$$

So the equation is $y = 2x + b$.

To find *b*, replace *x* and *y* by the *x*- and *y*-coordinates of either of the two given points, say (5, 5):

$$5 = 2(5) + b \Rightarrow 5 = 10 + b \Rightarrow b = -5$$

The equation is $y = 2x - 5$.

Any equation in which the only variables are *x* and *y*, and neither *x* nor *y* is written with an exponent, is the equation of a line.

EXAMPLE 9: $3y - 4x + 6 = 0$ is the equation of a line. To determine the slope and *y*-intercept of this line, first solve for y :

$$3y - 4x + 6 = 0 \Rightarrow 3y = 4x - 6 \Rightarrow y = \frac{4}{3}x - 2$$

The equation is now written in the form $y = mx + b$. We can see that the slope is $\frac{4}{3}$ and the y-intercept is –2.

In order to write the equation of a line, you need to know two things: the slope of the line and a point on the line. If the point you know happens to be the y-intercept, then, of course, you can use the equation $y = mx + b$. If the point is not the y-intercept, you have two choices. The first is to use the equation $y = mx + b$ and proceed as in Example 8. The second is to use the equation given in KEY FACT L9.

Key Fact L9

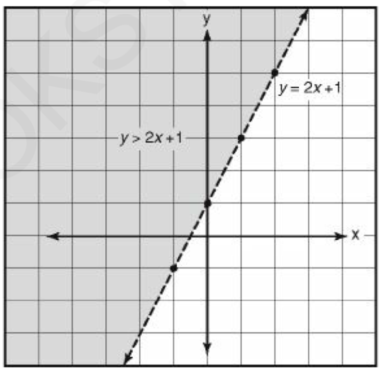
The equation of the line that passes through (x_1, y_1) and has slope m is $y - y_1 = m(x - x_1)$.

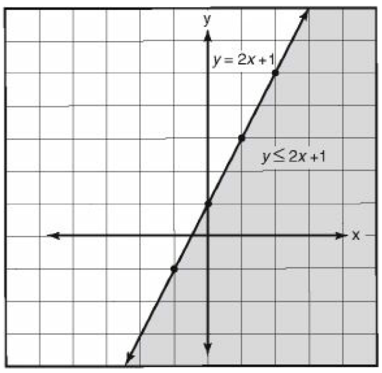
EXAMPLE 10: Line n passes through (1, 1) and is parallel to line ℓ, whose equation is $y = 2x - 3$. What is the equation of n?

Note that the slope of ℓ is 2, and since parallel lines have equal slopes, the slope of n is also 2. So the equation of n is:

$$y - 1 = 2(x - 1) \Rightarrow y - 1 = 2x - 2 \Rightarrow y = 2x - 1$$

Occasionally on the Math 1 test, there is a question concerning a linear inequality, such as $y > 2x + 1$. The graph of this inequality consists of all the points that are above the line $y = 2x + 1$. Note that the point (2, 5) is on the line, whereas (2, 6), (2, 7), and (2, 8) are all above the line. We indicate the set of all points satisfying the inequality by shading or striping the region above the line. To indicate that the points on the line itself do not satisfy the inequality, we draw a dotted line. To indicate that the points on a line are included in a graph, we draw a solid line. For example, (2, 5) is not on the graph of $y > 2x + 1$, but it is on the graph of $y \geq 2x + 1$. Similarly, the graph of $y < 2x + 1$ and $y \leq 2x + 1$ are shaded or striped regions below the line $y = 2x + 1$. These inequalities are shown in the following graphs.





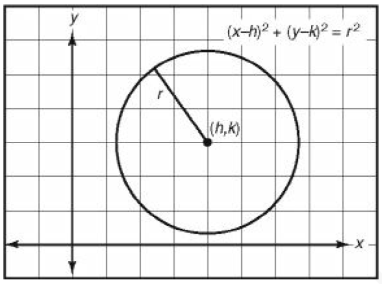
<u>CIRCLES AND PARABOLAS</u>

Besides the line, you need to know the equations for two other geometric shapes for the Math 1 test: the circle and the parabola.

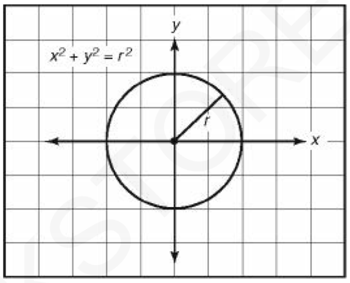
In this chapter, we review the standard equation of a circle, which is given in KEY FACT L10.

Key Fact L10

• **The equation of the circle whose center is the point (h, k) and whose radius is r is $(x-h)^2 + (y-k)^2 = r^2$.**



• **If the center is at the origin, (0, 0), then $h = k = 0$ and the equation reduces to $x^2 + y^2 = r^2$.**



To write the equation of a circle, you need to know its center and its radius. Either they will be specifically given to you or you will be given some other information that will enable you to determine them.

EXAMPLE 11: What is the equation of the circle whose center is at (–3, 2) and whose radius is 5?

Plug $h = -3$, $k = 2$, and $r = 5$ into the standard equation:

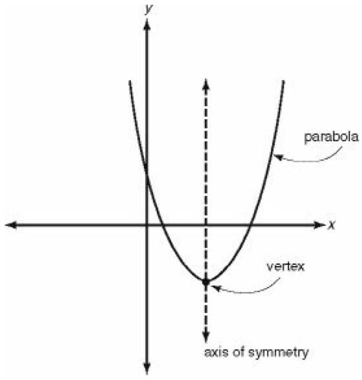
$$(x-(-3))^2 + (y-2)^2 = 5^2 \Rightarrow (x+3)^2 + (y-2)^2 = 25$$

EXAMPLE 12: In Examples 3 and 4, we considered the circle in which $C(3, -4)$ and $D(7, 2)$ are the endpoints of a diameter. What is the equation of this circle?

In Example 3, we used the midpoint formula to find that the center is at (5, –1). In Example 4, we used the distance formula to get that the radius is $\sqrt{13}$. Therefore, the equation for this circle is:

$$(x-5)^2 + (y-(-1))^2 = \left(\sqrt{13}\right)^2 \Rightarrow (x-5)^2 + (y+1)^2 = 13$$

There are many facts about ***parabolas*** that you do *not* need for the Math 1 test. In particular, you do not need to know the precise definition of parabola in terms of its *focus* and *directrix*. Basically, you need to know the general equation of a parabola and to recognize that the graph of a parabola is a U-shaped curve that is symmetrical about a line, called the ***axis of symmetry***, which passes through the parabola's ***vertex***, or ***turning point***. Any parabola you see on a <u>Math 1 test</u> will likely have a vertical axis of symmetry.

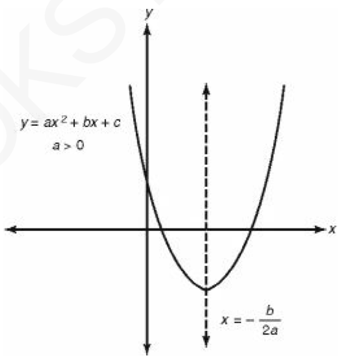


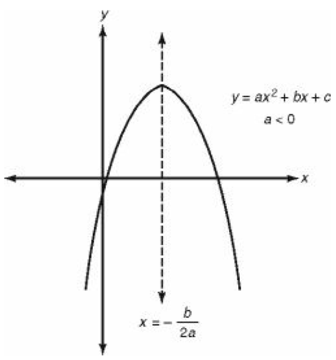
KEY FACT L11 lists the equations you need to know about parabolas.

Key Fact L11

For any real numbers a, b, c with $a \neq 0$:

- **$y = ax^2 + bx + c$ is the equation of a parabola whose axis of symmetry is a vertical line.**
- **Conversely, the equation of any parabola with a vertical axis of symmetry has an equation of the form $y = ax^2 + bx + c$.**
- **The equation of the parabola's axis of symmetry is $x = \frac{-b}{2a}$.**
- **The vertex of the parabola is the point on the parabola whose x-coordinate is $x = \frac{-b}{2a}$.**
- **If $a > 0$, the parabola opens upward and the vertex is the lowest point on the parabola.**
- **If $a < 0$, the parabola opens downward and the vertex is the highest point on the parabola.**





EXAMPLE 13: To determine the vertex (turning point) of the parabola $y = 2x^2 - 4x + 5$, first find its axis of

symmetry. Since $a = 2$, $b = –4$, and $c = 5$, the equation of the axis of symmetry is
. So the x-coordinate of the turning point is 1 and the y-coordinate is

$$2(1)^2 – 4(1) + 5 = 2 – 4 + 5 = 3$$

The turning point is (1, 3).

Of course, an alternative solution would be to graph $y = 2x^2 – 4x + 5$ on a graphing calculator and to trace along the parabola until the cursor is at the turning point.

Key Fact L12

Given any three points, *P*, *Q*, and *R* that do not lie on a line, there is a parabola with a vertical axis of symmetry that passes through them.

EXAMPLE 14: To find the equation of a parabola that passes through (0, 4), (1, 3), and (2, 6), plug the x and y coordinates of the points into the equation $y = ax^2 + bx + c$.

$x = 0$ and $y = 4$:

$$4 = a(0)^2 + b(0) + c \Rightarrow 4 = c$$

$x = 1$ and $y = 3$:

$$3 = a(1)^2 + b(1) + c = a + b + 4 \Rightarrow a + b = -1$$

$x = 2$ and $y = 6$:

$$6 = a(2)^2 + b(2) + c = 4a + 2b + 4 \Rightarrow$$
$$4a + 2b = 2 \Rightarrow 2a + b = 1$$

Subtract these last two equations:

Then by replacing a by 2 in the equation $a + b = –1$, we get that $b = –3$.

So the equation of the parabola that passes through the three given points is $y = 2x^2 – 3x + 4$.

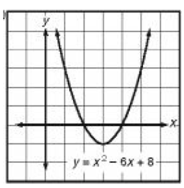
There is an important relationship between the parabola $y = ax^2 + bx + c$ and the quadratic equation $ax^2 + bx + c = 0$.

Key Fact L13

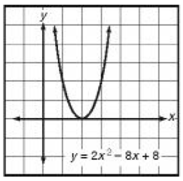
The *x*-intercepts of the graph of the parabola $y = ax^2 + bx + c$ are the (real) solutions of the equation $ax^2 + bx + c = 0$.

Consider the graphs of the following six parabolas.

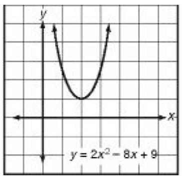
(i)



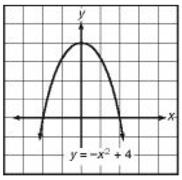
(ii)



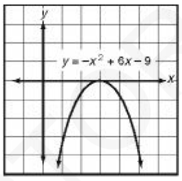
(iii)



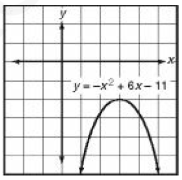
(iv)



(v)



(vi)



I. Since the graph of $y = x^2 - 6x + 8$ crosses the x–axis at 2 and 4, the quadratic equation $x^2 - 6x + 8 = 0$ has two real solutions: $x = 2$ and $x = 4$.

II. Since the graph of $y = 2x^2 - 8x + 8$ crosses the x-axis only at 2, the quadratic equation $2x^2 - 8x + 8 = 0$ has only one real solution: $x = 2$.

III. Since the graph of $y = 2x^2 - 8x + 9$ does not cross the x-axis, the quadratic equation $2x^2 - 8x + 9 = 0$ has no real solutions.

Similarly, from graphs iv, v, and vi, you can see that the equation $-x^2 + 4 = 0$ has two real solutions; the equation $-x^2 + 6x - 9 = 0$ has only one real solution; and the equation $-x^2 + 6x - 11 = 0$ has no real solutions.

Exercises

1. What is the equation of the line that passes through (3, –3) and (3, 3)?

(A) $x = 3$

(B) $y = 3$

(C) $y = 3x$

(D) $y = 3x + 3$

(E) $y = 3x - 3$

2. What is the equation of the line whose *x*- and *y*-intercepts are each 3?

(A) $x = 3$

(B) $y = 3$

(C) $y = x + 3$

(D) $y = x - 3$

(E) $y = -x + 3$

3. What is the slope of the line that passes through (4, 3) and is parallel to the line that passes through (–3, 4) and (3, –4)?

(A) $-\frac{4}{3}$

(B)

(C) $\frac{3}{4}$

(D) 1

(E) $\frac{4}{3}$

4. What is the slope of the line that passes through (4, 3) and is perpendicular to the line that passes through (–3, 4) and (3, –4)?

(A) $-\frac{4}{3}$

(B)

(C) $\frac{3}{4}$

(D) 1

(E) $\frac{4}{3}$

5. If *A* (–1, 0) and *B* (3, –2) are two adjacent vertices of square *ABCD*, what is the area of the square?

(A) 12

(B) 16

(C) 20

(D) 25

(E) 36

Questions 6 and 7 refer to circle *O*, in which *A* (–1, 0) and *B* (3, –2) are the endpoints of a diameter.

6. What is the area of circle *O*?

(A) 2.5π

(B) 5π

(C) 6.25π

(D) 10π

(E) 20π

7. Which of the following is the equation of circle *O*?

(A) $(x + 1)^2 + (y - 1)^2 =$

(B) $(x - 1)^2 + (y + 1)^2 =$

(C) $(x + 1)^2 + (y - 1)^2 = 5$

(D) $(x - 1)^2 + (y + 1)^2 = 5$

(E) $(x - 1)^2 + (y + 1)^2 = 25$

8. If *M* (3, 1) and *N* (7, 1) are two adjacent vertices of a rectangle, which of the following could *not* be one of the rectangle's other vertices?

(A) (3, 7)

(B) (7, 3)

(C) (3, –7)

(D) (–3, 7)

(E) (7, 7)

9. Which of the following is the equation of a parabola that does NOT intersect the *x*-axis?

(A) $y = x^4 + 1$

(B) $y = x^2 + 2x - 3$

(C) $y = (x - 1)^2$

(D) $y = x^2 - 1$

(E) $y = x^2 + 1$

10. What is the slope of the line that passes through (*a*, *b*) and $\left(\frac{1}{a}, b\right)$?

(A) 0

(B) $\frac{1}{b}$

(C) $\frac{1-a^2}{a}$

(D) $\frac{a^2-1}{a}$

(E) The slope is undefined.

11. Line ℓ is tangent to the circle whose center is at (3, 2). If the point of tangency is (6, 6), what is the slope of line

ℓ?

(A) $-\frac{4}{3}$

(B) $-\frac{3}{4}$

(C) 0

(D) $\frac{3}{4}$

(E) $\frac{4}{3}$

Questions 12–14 concern the parabola whose equation is $y = x^2 - 20x - 69$.

12. Where does the graph of the parabola cross the *y*-axis?

(A) –69

(B) –3

(C) 23

(D) 69

(E) The graph does not cross the *y*-axis.

13. What is the sum of the *x*=coordinates of the points where the graph of the parabola crosses the *x*-axis?

(A) –69

(B) 0

(C) 20

(D) 69

(E) The graph does not cross the *x*-axis.

14. Which of the following is the parabola's turning point?

(A) (–10, 231)

(B) (0, –69)

(C) (10, –169)

(D) (-3, 0)

(E) (23, 0)

15. What is the equation of the circle whose center is at the origin and that passes through the point (8, 15)?

(A) $x^2 + y^2 = 15$

(B) $x^2 + y^2 = 17$

(C) $x^2 + y^2 = 289$

(D) $(x - 8)^2 + (y - 15)^2 = 17$

(E) $(x - 8)^2 + (y - 15)^2 = 289$

ANSWERS EXPLAINED

Answer Key

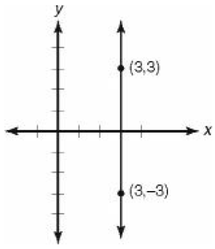
1. **(A)**	6. **(B)**	11. **(B)**
2. **(E)**	7. **(D)**	12. **(A)**
3. **(A)**	8. **(D)**	13. **(C)**
4. **(C)**	9. **(E)**	14. **(C)**
5. **(C)**	10. **(A)**	15. **(C)**

Solutions

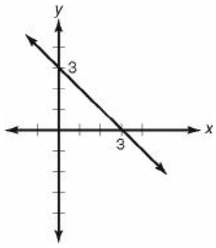
Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the "Tactics" chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(A)** A quick sketch shows that the line that passes through (3, –3) and (3, 3) is vertical. By KEY FACT L8, its equation is $x = 3$.



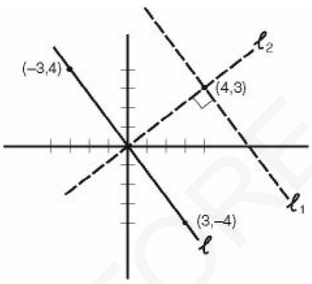
2. **(E)** Quickly sketch the line.



Since the line is neither horizontal nor vertical, its equation has the form $y = mx + b$. Since it crosses the y-axis at (0, 3), $b = 3$.

Since it passes through (0, 3) and (3, 0), its slope is $\frac{0-3}{3-0} = \frac{-3}{3} = -1$, and so $y = -1x + 3$ or $y = -x + 3$.

Solutions 3 and 4. Before answering either question, quickly sketch line ℓ that goes through (–3, 4) and (3, –4) and lines ℓ_1 and ℓ_2, that go through (4, 3) and are parallel to ℓ and perpendicular to ℓ, respectively.



You should immediately see that the slope of ℓ_1 is negative and the slope of ℓ_2 is positive. So the answer to Question 3 must be A or B, and the answer to Question 4 must be C, D, or E.

3. **(A)** By KEY FACT L5, the slope of ℓ, the line through (–3, 4) and (3, –4) is

$$\frac{-4-4}{3-(-3)}=\frac{-8}{6}=-\frac{4}{3}$$

By KEY FACT L7, parallel lines have equal slopes, so the slope of ℓ_1 is $-\frac{4}{3}$. Note that it is irrelevant that ℓ_1 passes through (4, 3).

4. **(C)** From Solution 3, above, the slope of ℓ is $-\frac{4}{3}$. By KEY FACT L7, if two lines are perpendicular, the product of their slopes is –1. So if *m* is the slope of ℓ_2, then:

$$-\frac{4}{3}m = -1 \Rightarrow -4m = -3 \Rightarrow m = \frac{3}{4}$$

As in Question 3, it is irrelevant that ℓ_2 passes through (4, 3).

5. **(C)** From the distance formula (KEY FACT L3):

$$AB = \sqrt{3-(-1)^2 + (-2-0)^2} = \sqrt{4^2 + (-2)^2}$$
$$= \sqrt{16+4} = \sqrt{20}$$

The area of square $ABCD = (AB)^2 = \left(\sqrt{20}\right)^2 = 20.$

6. **(B)** By the distance formula (KEY FACT L3):

$$AB = \sqrt{(3-(-1))^2 + (-2-0)^2} = \sqrt{4^2 + (-2)^2}$$
$$= \sqrt{16+4} = \sqrt{20}$$

Since the diameter is $\sqrt{20}$, the radius is $\frac{\sqrt{20}}{2}$. Now use the area formula:

$$A = \pi r^2 = \pi\left(\frac{\sqrt{20}}{2}\right)^2 = \pi\left(\frac{20}{4}\right) = 5\pi$$

**Alternatively, you could have used the midpoint formula (KEY FACT L4) to determine that the center *O* is at

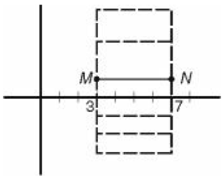
$$\left(\frac{-1+3}{2}, \frac{0+(-2)}{2}\right) = \left(\frac{2}{2}, \frac{-2}{2}\right) = (1, -1)$$

and then used the distance formula to find the length of radius $\overline{OA}$.

7. **(D)** From the solution to Question 6, above, the radius of circle *O* is $\frac{\sqrt{20}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$. From the alternative solution to Question 6, the center is at (1, –1). Finally, by KEY FACT J11, the equation of circle *O* is:

$$(x-1)^2 + (y-(-1))^2 = (\sqrt{5})^2$$
$$(x-1)^2 + (y+1)^2 = 5$$

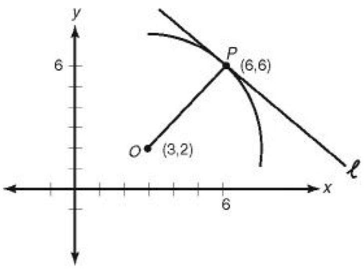
8. **(D)** Any point whose *x*-coordinate is 3 or 7 could be another vertex. Of the choices, only (–3, 7) could not be one of the vertices.



9. **(E)** A graph intersects the *x*-axis at points whose *y*-coordinates are 0. Choices B, C, and D all intersect the *x*-axis at (1, 0). Choices A and E do not intersect the *x*-axis, but Choice A is not the equation of a parabola.

10 **(A)** By KEY FACT L5, the formula for slope is $\frac{y_2 - y_1}{x_2 - x_1}$. Before using it, though, look. Since the *y*-coordinates are equal, the numerator, and thus the fraction, equals 0.

11. (**B**) Draw a rough sketch



Line segment $\overline{OP}$, joining (3, 2) and (6, 6) is a radius and so by KEY FACT L10 is perpendicular to line ℓ. The slope of $\overline{OP}$ is $\frac{6-2}{6-3}=\frac{4}{3}$. Therefore, by -34 KEY FACT L7, the slope of ℓ is $-\frac{3}{4}$.

12. **(A)** Since the *y*-intercept of a graph is a point whose *x*-coordinate is 0, just evaluate the equation when $x = 0$:

$$y = 0^2 - 20(0) - 69 = -69$$

13. **(C)** The x-intercepts of the graph are the solutions of the equation $x^2 - 20x - 69 = 0$.

$$x^2 - 20x - 69 = 0 \Rightarrow (x - 23)(x + 3) = 0 \Rightarrow$$
$$x = 23 \text{ or } x = -3$$

The sum of the roots is 23 + (–3) = 20.

Since the question asks for the sum of the roots and not the individual roots, you can use KEY FACT E3—the formula for the sum of the roots:

$$\text{sum of the roots} = \frac{-b}{a} = \frac{-(-20)}{1} = 20$$

14. **(C)** The turning point of the parabola is on the parabola's axis of symmetry, whose equation is $x = \frac{-b}{2a} = \frac{20}{2} = 10$. So the x-coordinate of the turning point is 10; to get the y-coordinate, replace x by 10 in the equation of the parabola:

$$y = (10)^2 - 20(10) - 69 = 100 - 200 - 69 = -169.$$

The turning point is (10, –169).

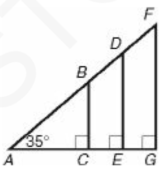
 15. **(C)** The equation of a circle whose center is at the origin is $x^2 + y^2 = r^2$, where r is the radius. Since (8, 15) is on the circle, $r^2 = 8^2 + 15^2 = 64 + 225 = 289$. So the equation of this circle is $x^2 + y^2 = 289$.

TRIGONOMETRY

Basic Trigonometry

This chapter is very short, because it does not review all the trigonometry you learned in school. It reviews only the trigonometry you need for the Math 1 test—which is not very much. The ONLY trigonometry you need to know for the Math 1 test are the meanings of the trigonometric ratios—sine, cosine, and tangent— and one simple identity.

SINE, COSINE, AND TANGENT



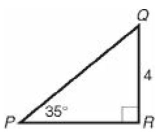
In the figure above, right triangles *ABC*, *ADE*, and *AFG* each have a 90° angle and a 35° angle, so they are all similar to one another. Therefore, their sides are in proportion:

$$\frac{BC}{AB}=\frac{DE}{AD}=\frac{FG}{AF}=\frac{\text{length of the side opposite the } 35^\circ \text{ angle}}{\text{length of the hypotenuse}}$$

This ratio is called the sine of 35° and is written sin 35°. To evaluate sin 35°, you could very carefully measure the lengths of $\overline{FG}$ and $\overline{AF}$ and divide. In the given figure, $FG \approx 1$ inch, $AF \approx 1.75$ inches, so $\frac{FG}{AF} \approx \frac{1}{1.75} = 0.57$.

Fortunately, you do not have to do this. You can use your calculator. Depending on what calculator you use, you would either enter 35 and then press the [sin] button or press the [sin] button and then enter 35. Regardless, in a fraction of a second, you will see the answer correct to several decimal places: sin 35° = 0.573576436, far greater accuracy than you need for the Math 1 test.

EXAMPLE 1: To find the length of hypotenuse $\overline{PQ}$ in the triangle below, use the sine ratio (and your calculator):



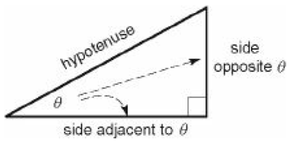
$$\sin 35° = \frac{4}{PQ} \Rightarrow (PQ)\sin 35° = 4 \Rightarrow$$

$$PQ = \frac{4}{\sin 35°} = \frac{4}{0.574} = 6.97$$

The formal definitions of the three trigonometric ratios you need are given in KEY FACT M1.

Key Fact M1

Let θ be one of the acute angles in a right triangle.



- **The formula for the sine of θ , denoted sinθ, is:**

$$\sin\theta = \frac{\text{the length of the side opposite } \theta}{\text{the length of the hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}$$

- **The formula for the cosine of θ , denoted cosθ, is:**

$$\cos\theta = \frac{\text{the length of the side adjacent to } \theta}{\text{the length of the hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

- **The formula for the tangent of θ , denoted tanθ, is:**

$$\tan\theta = \frac{\text{the length of the side opposite } \theta}{\text{the length of the side adjacent to } \theta} = \frac{\text{opposite}}{\text{adjacent}}$$

- **From the definitions of the three trigonometry ratios, it follows immediately that for any acute angle θ,** $\tan\theta = \frac{\sin\theta}{\cos\theta}$

For decades, students have remembered these definitions by memorizing the "word" SOHCAHTOA. For example, the "S" in "Soh" stands for "sine" and the "OH" reminds you that sine is Opposite over Hypotenuse.

The only trigonometric equation or identity you need to know other than those in KEY FACT M1, for the Math 1 test, is given in KEY FACT M2.

Key Fact M2

For any angle θ , $\sin^2\theta + \cos^2\theta = 1$.

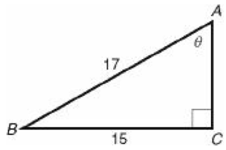
Note that $\sin^2\theta$ is an abbreviation for $(\sin\theta)^2$. So the identity $\sin^2\theta + \cos^2\theta = 1$ means that if you take the sine of any angle and square it and then take the cosine of that same angle and square it, the sum of those two squares is 1.

For example, you can use your calculator to verify that $(\sin 37°)^2 + (\cos 37°)^2 = 1$. Note that in this identity, θ can be replaced by any expression at all. So $\sin^2(4x + 1) + \cos^2(4x + 1) = 1$.

EXAMPLE 2: $(3\sin^2 3\theta + 3\cos^2 3\theta - 1)^2 = (3(\sin^2 3\theta + \cos^2 3\theta) - 1)^2 = (3(1) - 1)^2 = (3 - 1)^2 = 2^2 = 4$.

If you know the value of any one of sinθ, cosθ, or tanθ, you can always find the values of the other two.

EXAMPLE 3: If you are given that $\sin\theta = \frac{15}{17}$ and want to know the 17 value of cosθ or tanθ, draw right triangle *ABC* and label *BC*, the side opposite θ, as 15 and the hypotenuse as 17.



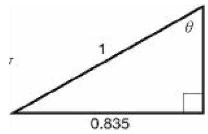
Now use the Pythagorean theorem to find *AC*:

$$(AC)^2 + 15^2 = 17^2 \Rightarrow AC^2 + 225 = 289 \Rightarrow$$
$$(AC)^2 = 64 \Rightarrow AC = 8.$$

So $\theta = \frac{8}{17}$ and $\tan\theta = \frac{15}{8}$

If you were given that sin θ = 0.835, you would proceed exactly the same way. Draw a triangle and use the Pythagorean theorem to get

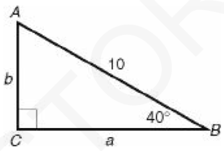
$x^2 + (0.835)^2 = 1^2 \Rightarrow x^2 + 0.697 = 1 \Rightarrow x^2 = 0.303 \Rightarrow x = 0.55.$



Now use SOHCAHTOA:

$\cos\theta = \frac{0.55}{1} = 0.55 =$ and $\tan\theta = \frac{0.835}{0.55} = 1.52$

EXAMPLE 4: To find the values of *a* and *b* in the triangle below, use the trigonometric ratios.



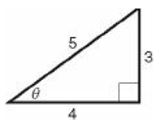
$$\sin 40° = \frac{b}{10} \Rightarrow b = 10(\sin 40°) = 10(0.643) = 6.43$$

$$\cos 40° = \frac{a}{10} \Rightarrow a = 10(\cos 40°) = 10(0.766) = 7.66$$

You could also be asked to find the measure of an angle θ if you are given the value of the sine, cosine, or tangent of that angle. For example, if you are given that sinq = 0.8, to find the value of θ, you use your calculator to evaluate $\sin^{-1}(0.8)$. On most calculators, SIN^{-1} is the second function button above the SIN button. So depending on your calculator, you would either press [2nd] [SIN] [.8] [enter] or [.8] [2nd] [SIN]

EXAMPLE 5: What is the measure of the smallest angle in a 3–4–5 triangle?

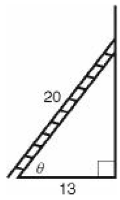
Draw and label the triangle, which you should immediately recognize as a right triangle.



By KEY FACT H3, the smallest angle is opposite the smallest side. So $\sin\theta = \frac{3}{5}$ and $\theta = \sin^{-1}\left(\frac{3}{5}\right) = 36.87°$. Note that you can convert $\frac{3}{5}$ to 0.6 and take $\sin^{-1}(0.6)$, but that isn't necessary.

EXAMPLE 6: A 20-foot ladder is leaning against a vertical wall. If the base of the ladder is 13 feet from the wall, what is the angle formed by the ladder and the ground?

Of course, you start by drawing a diagram.



Then:

$$\cos\theta = \frac{13}{20} \Rightarrow \theta = \cos^{-1}\left(\frac{13}{20}\right) = 49.46°$$

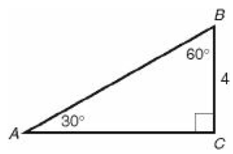
WHAT YOU *DON'T* NEED TO KNOW

You have just reviewed the ONLY trigonometry you need to know for the Math 1 test. The College Board is very specific about this. Here is a list of several topics in trigonometry that you may have studied in school and that may be tested on the Math Level 2 test but are NOT included on the Math Level 1 test:

- Radian measure (**so keep your calculator in degree mode**)
- Angles whose measures are greater than 90° or less than 0°
- Reference angles
- The reciprocal trigonometric functions: secant, cosecant, and cotangent
- Graphs of the trigonometric functions
- Amplitude and period of the trigonometric functions
- Inverse trigonometric functions
- The law of sines and the law of cosines
- The trigonometric formula for the areas of a triangle and parallelogram
- Double-angle formulas and half-angle formulas
- Trigonometric identities (other than the two previously mentioned)

There is no question on the Math 1 test whose correct solution requires you to use any topic in the preceding list. However, you may always use anything you know.

For example, in the figure below, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(4)$.



So in order to calculate the area, you need to know the value of *AC*. Since $\triangle ABC$ is a 30-60-90 right triangle, by KEY FACT H8, $AC = 4\sqrt{3}$.

You can also find *AC* by using the tangent ratio: tan 30° = $\frac{4}{AC}$ and solving for *AC*. If you know the law of sines and for any reason you prefer to use it, of course you may:

$$\frac{4}{\sin 30°} = \frac{AC}{\sin 60°} \Rightarrow AC = \frac{4 \sin 60°}{\sin 30°} = 6.93$$

Exercises

1. In the triangle below, what is the sum of *a* and *b* ?

(A) 1.28

(B) 5

(C) 6.41

(D) 7.17

(E) 8.45

2. If 0° < θ < 90°, then which of the following is an expression for $\tan^2\theta\cos^2\theta + \cos^2\theta$?

(A) 0

(B) 1

(C) $\sin^2\theta$

(D) $\tan^2\theta + 1$

(E) $(\tan\theta\cos\theta + \cos\theta)^2$

3. A ladder is leaning against a wall, forming an angle of 65° with the ground. If the foot of the ladder is 8 feet from the wall, what is the length of the ladder, in feet?

(A) 3.38

(B) 7.25

(C) 8.83

(D) 17.15

(E) 18.93

4. If 0° < θ < 90° and $\tan\theta = 5$, what is $\sin\theta + \cos\theta$?

(A) 0.19

(B) 0.85

(C) 1

(D) 1.18

(E) 5.2

5. If 0° < θ < 90°, then which of the following is equivalent to $\frac{\cos\theta}{\tan\theta} + \sin\theta$?

(A) 1

(B) $\frac{1}{\sin\theta}$

(C) $1 + \tan\theta$

(D) $\frac{\cos^2\theta}{\sin^2\theta}$

(E) cos θ + tan θsin θ

6. What is the measure of the smallest angle in a 5=12=13 right triangle?

(A) 15°

(B) 21°

(C) 22.6°

(D) 45°

(E) 67.4°

7. If ΔABC is isosceles and m$\angle C$ = 90°, which of the following must be true?

I. tan A = 1

II. sin A = cos A

III. sin A = cos B

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I, II, and III

8. Which of the following is equal to (3 sin2 θ)(4 sin2 θ) + (2 cos2 θ)(6 cos2 θ)?

(A) 1

(B) 2

(C) 6

(D) 12

(E) 24

9. A kite string is tied to a peg in the ground. If the angle formed by the string and the ground is 70° and if there is 100 feet of string out, to the nearest foot, how high above the ground is the kite?

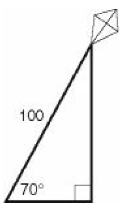
(A) 34

(B) 64

(C) 74

(D) 94

(E) 274



10. If $0° < \theta < 90°$, which of the following is equivalent to $\frac{\sin^4\theta - \cos^4\theta}{\cos^2\theta - \sin^2\theta}$?

(A) –1

(B) 1

(C) $\sin^2\theta - \cos^2\theta$

(D) $\cos^2\theta - \sin^2\theta$

(E) $\tan^2\theta - \cos^2\theta$

11. If the longer leg of a right triangle is twice as long as the shorter leg, what is the ratio of the measure of the larger acute angle to the measure of the smaller acute angle?

(A) 0.42

(B) 0.50

(C) 1.78

(D) 2.00

(E) 2.39

12. A car is parked 120 feet from a building that is 350 feet tall. What is the measure of the angle of depression from the top of the building to the car?

(A) 18.9°

(B) 37.8°

(C) 64.2°

(D) 71.1°

(E) 78.6°

ANSWERS EXPLAINED

Answer Key

1. **(C)**	5. **(B)**	9. **(D)**
2. **(B)**	6. **(C)**	10. **(A)**
3. **(E)**	7. **(E)**	11. **(E)**
4. **(D)**	8. **(D)**	12. **(D)**

SOLUTIONS

Each of the problems in this set of exercises is typical of a question you could see on a Math 1 test. When you take the model tests in this book and, in particular, when you take the actual Math 1 test, if you get stuck on questions such as these, you do not have to leave them out—you can almost always answer them by using one or more of the strategies discussed in the "Tactics" chapter. The solutions given here do *not* depend on those strategies; they are the correct mathematical ones.

See Important Tactics for an explanation of the symbol ⇒, which is used in several answer explanations.

1. **(C)** Use the sine and cosine ratios:

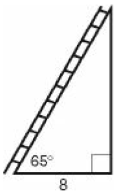
$$\sin 20° = \frac{a}{5} \Rightarrow a = 5(\sin 20°) = 5(0.342) = 1.710$$

$$\cos 20° = \frac{b}{5} \Rightarrow b = 5(\cos 20°) = 5(0.940) = 4.700$$

So $a + b = 1.710 + 4.700 = 6.41$

2. **(B)** $\tan^2\theta\cos^2\theta + \cos^2\theta = \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \cancel{\cos^2\theta} + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$

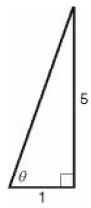
3. **(E)** First draw a diagram.



By using the cosine ratio, we get that cos 65° = , where L represents the length of the ladder.

L cos 65° = 8 ⇒
L = 8 ÷ (cos 65°) = 18.93

4. **(D)** Draw a right triangle. Since $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$, label the side opposite θ as 5 and the side adjacent to θ as 1.



Then use the Pythagorean theorem to find the hypotenuse:

$$1^2+5^2=c^2 \Rightarrow c^2=1+25=26 \Rightarrow c=\sqrt{26}$$

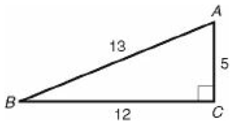
So $\sin\theta = \frac{5}{\sqrt{26}}$ and $\cos\theta = \frac{1}{\sqrt{26}}$ Then:

$$\sin\theta + \cos\theta = \frac{5}{\sqrt{26}} + \frac{1}{\sqrt{26}} = \frac{6}{\sqrt{26}} = 1.18$$

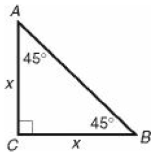
Alternatively, you could have found that $\theta = \tan^{-1}(5) = 78.69°$ and then added sin78.69 + cos78.69 = 1.18.

5. **(B)** $\frac{\cos\theta}{\tan\theta} + \sin\theta = \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}} + \sin\theta = \frac{\cos^2\theta}{\sin\theta} + \sin\theta = \frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta} = \frac{1}{\sin\theta}$

6. **(C)** Draw and label a 5-12-13 right triangle. The smallest angle is *B*, the angle opposite the smallest side. Since $\sin B = \frac{5}{13}$, we have $B = \sin^{-1}\left(\frac{5}{13}\right) = 22.6°$.



7. **(E)** Sketch isosceles right ΔABC with $AC = BC$ and $m\angle A = m\angle B = 45°$.



Then $\tan A = \frac{x}{x} = 1$ (I is true). $\sin A$, $\cos A$, $\sin B$, and $\cos B$ are all equal to $\frac{x}{AB}$ (II and III are true). Note that $\sin 45° = \cos 45° = 0.707$.

8. **(D)** $(3 \sin 2\theta)(4 \sin 2\theta) = 12 \sin^2 2\theta$

$(2 \cos 2\theta)(6 \cos 2\theta) = 12 \cos^2 2\theta$

So, $(3 \sin 2\theta)(4 \sin 2\theta) + (2 \cos 2\theta)(6 \cos 2\theta) = 12 \sin 2\,^2\theta + 12 \cos^2 2\theta = 12(\sin^2 2\theta + \cos^2 2\theta) = 12(1) = 12$

9. **(D)** If *x* represents the height of the kite, then

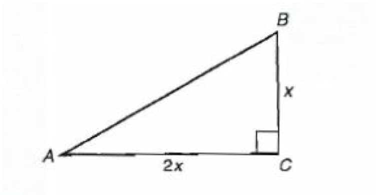
$$\sin 70° = \frac{x}{100} \Rightarrow x = 100 \sin 70° = 100(0.9396) = 93.96 \approx 94.$$

10. **(A)** Since $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$:

$$\sin^4\theta - \cos^4\theta = (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$$
$$= (\sin^2\theta - \cos^2\theta)(1) = \sin^2\theta - \cos^2\theta$$

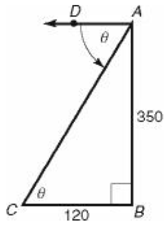
$$\frac{\sin^4\theta - \cos^4\theta}{\cos^2\theta - \sin^2\theta} = \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta - \sin^2\theta}$$
$$= \frac{\sin^2\theta - \cos^2\theta}{-\left(\sin^2\theta - \cos^2\theta\right)} = -1$$

11. **(E)** Draw and label a diagram. Then:



$$\tan B = \frac{2x}{x} = 2 \Rightarrow \mathbf{m}\angle B = \tan^{-1}(2) = 63.435°$$

12. **(D)** Draw a diagram and label it.



The angle of depression, ∠*DAC*, is the angle between the line of sight and the horizontal and by KEY FACT G6 is congruent to ∠*C*. Since

Data Analysis, Statistics, and Probability

Data Analysis, Statistics, and Probability

Data Analysis and Statistics

MEASURES AND REGRESSION

Quantitative data are number sets such as heights, weights, test scores, tensile strength, and so forth. By contrast, *categorical* data consist of descriptive labels, such as hair color, city of residence, socioeconomic status, and the like. Since the Math Level 2 test is unlikely to include questions about categorical data, the concepts described below pertain to quantitative data only.

Measures of center summarize a data set using a single “typical” value. Three measures of center might be encountered on the Math Level 2 test: mean, median, and mode.

The *mean* is the sum of all the data values divided by the number of values. The formula for the mean $\bar{x}$ of a data set is $\bar{x} = \frac{\sum x_i}{n}$, where Σ indicates the sum of the data values x_i and n is the number of data values.

To determine the *median* , the data must first be ordered. If the number of values is odd, the median is the single middle value. If the number of values is even, the median is the mean of the two middle values. There is no formula for the median of a data set.

The *mode* is the value that appears most often. There is no formula for the mode of a data set.

EXAMPLE 1

The heights of the starting basketball team for South High School are 69”, 72”, 75”, 78”, and 78”. Find the mean, median, and mode of this data set.

The mean is $\frac{69+72+75+78+78}{5} = 74.4$". The median is 75". The mode is 78".

EXAMPLE 2

The mean of 24 test scores is 77.5. When the 25th class member takes the test, the mean goes down by 1.1 points. What was that 25th score?

The total of the 24 test scores is 24 × 77.5 = 1860, and the total of the 25 test scores is 25 × 76.4 = 1910. Therefore, the 25th score is 1910 – 1860 = 50.

EXAMPLE 3

What is the median of the frequency distribution shown in the table?

Data Value	**Frequency**
24	**3**
25	**7**
26	**5**
27	**1**

There are 16 data values altogether, so the median is the mean of the 8th and 9th largest values. Both of these values are 25, so the median is also 25.

The Math Level 2 test might ask questions about **measures of spread**. These quesitons ask about how spread out a set of data values is.

The *range* is a measure of spread. It is the difference between the largest and smallest data values.

EXAMPLE 4

Find the range of the data values 85, 96, 72, 89, 66, and 78.

The largest value is 96 and the smallest is 66. The range is 96 – 66 = 30.

Loosely speaking, the *standard deviation* is the "average" difference between individual data values and their mean. The formula for the standard deviation s of a data set is $s = \sqrt{\frac{1}{n-1}\sum(x_i - \overline{x})^2}$. The larger the standard deviation, the more spread out a data set is. Standard deviation is a unit-free measure of the "distance" between a specific data value and the mean. Thus the standard deviation can be used to compare single data values from different data sets. A z-score, where $z = \frac{x - \overline{x}}{s}$, is the number of standard deviations s that a data value x is from the mean $\overline{x}$. The greater the value of $|z|$, the less common the data value x is. In other words, fewer data values have a high z-score.

EXAMPLE 5

Which data set has the smaller standard deviation: {5, 7, 9} or {4, 7, 10}?

Both data sets have a mean of 7. However, the first set is less spread out than the second, so the first has the smaller standard deviation. According to the formula, the standard deviation of the first data set is 2 while that of the second data set is 3.

EXAMPLE 6

A chart showing sports statistics for a particular school is shown below. Which is statistically a better score: 50.30 seconds in the backstroke or 74 inches in the high jump?

Stroke	**Mean**	**Standard Deviation**
Backstroke	**50.72 sec.**	**0.24 sec.**
High Jump	**72.9 in.**	**0.54 in.**

A time of 50.30 seconds in the backstroke is $z = \frac{50.30 - 50.72}{0.24} = -1.75$ standard deviations better (less) than the backstroke mean. A height of 74 inches in the high jump is $z = \frac{74 - 72.9}{0.54} = 2.04$ standard deviations better (more) than the high jump mean. Therefore, the high jump performance is better.

Measures of center and spread apply to a single variable. *Regression* is a technique for analyzing the relationship between two variables. This technique summarizes relationships such as mathematical equations in which the two variables are denoted by x (the independent variable) and y (the dependent variable). The Math Level 2 test may ask about any one of three models to capture the relationship between x and y :

- Linear model $y = a_0 + a_1x$
- Quadratic model $y = a_0 + a_1x + a_2x^2$
- Exponential model $y = a_0e^{a_1x}$

The figures below show scatter plots having these shapes. Regression techniques use paired values (x , y) to estimate *parameter* values a_0, a_1, a_2, depending on the model selected. Once this is done, the equation for that model can be used to predict y for a given value of x.

The Level 2 test does not require students to know the mathematics of regression techniques. Students should know how to use their calculators to get parameter estimates for a particular model and to use the equation as a prediction tool.

EXAMPLE 7

The decennial population of Center City for the past five decades is shown in the table below. Use exponential regression to estimate the 1965 population.

Population of Center City

Year	***Population***
1950	**48,000**
1960	**72,000**
1970	**95,000**
1980	**123,000**
1990	**165,000**

Transform the years to "number of years after 1950" and enter these values into L4. Then enter the populations in thousands. Set up the scatter plot by pressing 2ndY= and selecting a plot (Plot 1). Turn the plot on, select the scatter plot logo, and enter the list names. Then press STAT/CALC/ExpReg L4,L5,Y1. This will store the regression equation in Y1. The resulting command is shown in the left screen below. Press ENTER to display the values for the equation. These are shown in the right screen below.

Press ZOOM/9 to view the scatterplot and exponential curve. Press 2nd/CALC/value and enter 15, representing 1965. The cursor moves to the point on the regression curve where $x = 15$ and displays both x and y at the bottom of the screen, as shown below.

The 1965 population was about 79,300.

EXERCISES

1. Last week, police ticketed 13 men traveling 18 miles per hour over the speed limit and 8 women traveling 14 miles per hour over the speed limit. What was the mean speed over the limit of all 21 drivers?
   (A) 16 miles per hour
   (B) 16.5 miles per hour
   (C) 17 miles per hour
   (D) none of these
   (E) cannot be determined
2. If the range of a set of integers is 2 and the mean is 50, which of the following statements must be true?
   I. The mode is 50
   II. The median is 50
   III. There are exactly three data values

(A) only I
(B) only II
(C) only III
(D) I and II
(E) I, II, and III

3. What is the median of the frequency distribution shown below?

Data Value	Frequency
0	1
1	3
2	7
3	15
4	10
5	7
6	3
7	3

(A) 2
(B) 3
(C) 4
(D) 5
(E) Cannot be determined

4. Which of the following statements must be true?
I. The range of a data set must be smaller than its standard deviation.
II. The standard deviation of a data set must be smaller than its mean.
III. The median of a data set must be smaller than its mode.

(A) I only
(B) I and II
(C) II only
(D) I, II, and III
(E) none are true

5. The mean and standard deviation for EST math scores are shown in the table below for five high schools in a large city. A particular score for each city is also shown (in the right column).

School	Mean	Standard Deviation	Single Score
A	532	24	600
B	485	30	560
C	515	22	561
D	396	26	474
E	479	35	552

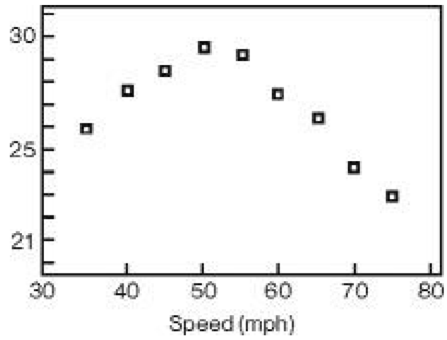
Which single score has the highest *z*-score?
(A) 474 in school D
(B) 552 in school E
(C) 560 in school B
(D) 561 in school C
(E) 600 in school A

6. Jack recorded the amount of time he studied the night before each of 4 history quizzes and the score he got on each quiz. The data are in the table below.

Score	Time (min.)
86	45
70	15
90	40
78	35

Use linear regression to estimate the score Jack would get if he studied for 20 minutes.
(A) 71
(B) 72
(C) 73
(D) 74
(E) 75

7. The scatter plot shows gas mileage (miles per gallon) at various speeds (miles per hour) when a car was driven 100 miles at various speeds on a test track.



Which regression model is probably the best predictor of gas mileage as a function of speed?

(A) constant
(B) linear
(C) quadratic
(D) cubic
(E) exponential

Answers and Explanations

Measures and Regression

1. **(B)** There are 13 eighteens and 8 fourteens, so the total over the speed limit is 346. Divide this by the 21 people to get 16.5.

2. **(B)** Since the data values are integers, the range is 2, and the mean is 50, the possible data values are 49, 50, and 51.

I. The set could consist of equal numbers of 49s and 51s and have a mean of 50 without 50 even being a data value. So I need not be true.
II. Since the mean is 50, there must be equal numbers of 49s and 51s, so 50 is also the median. II must be true.
III. Explanations in I and II imply that III need not be true.

3. **(B)** There are 49 data values altogether, so the median is the 25th largest. Adding the frequencies up to 25 puts the 25th number at 3.

4. **(E)** None are true. The range of any data set must be larger than its standard deviation because the range measures total spread while the standard deviation measures average spread. So Choice I is false. Either the mean or standard deviation of a data set can be larger. For example, the mean of the data {1, 5, 10} is 5.3, while its standard deviation is 4.51. The mean of the data set {1, 5, 20} is 8.7, while its standard deviation is 10.0. So Choice II is false. Either the median or mode of a data set can be larger. For example, the median of the data

set {1, 2, 3, 4, 4} is 3, while its mode is 4. The median of the data set {1, 1, 2, 3, 4} is 2 while its mode is 1. So Choice III is false.

5. **(D)** The z-scores for the five schools are 2.8 for A, 2.5 for B, 2.1 for C, 3 for D, and 2.1 for E.

6. **(C)** Enter the data in two lists (study times in L1 and test scores in L2). Enter STAT/CALC/8, and enter VARS/YVARS/Function/Y_1, followed by ENTER. This produces estimates of the slope (b) and y-intercept (a) of the regression line $a + bx$. Enter this expression into Y_1. Enter $Y_1(2)$ to get the score of 72.

7. **(C)** The scatter plot has the shape of a parabola with a maximum. Therefore, the quadratic model would be the best predictor.

4.2 Probability

The probability of an event happening is a number defined to be the number of ways the event can happen successfully divided by the total number of ways the event can happen.

EXAMPLE 1

What is the probability of getting a head when a coin is flipped?

A coin can fall in one of two ways, heads or tails. The two are equally likely.

$$P(\text{head}) = \frac{\text{number of ways a head can come up}}{\text{total number of ways the coin can fall}} = \frac{1}{2}.$$

EXAMPLE 2

What is the probability of getting a 3 when one die is thrown?

A die can fall with any one of six different numbers showing, and there is only one way a 3 can show.

$$P(3) = \frac{\text{number of ways a 3 can come up}}{\text{total number of ways the die can fall}} = \frac{1}{6}.$$

EXAMPLE 3

What is the probability of getting a sum of 7 when two dice are thrown?

Since it is not obvious how many different throws will produce a sum of 7, or how many different ways the two dice will land, it will be useful to consider all the possible outcomes. The set of all outcomes of an experiment is called the *sample space* of the experiment. In order to keep track of the elements of the sample space in this experiment, let the first die be green and the second die be red. Since the green die can fall in one of six ways, and the red die can fall in one of six ways, there should be $6 \cdot 6$ or 36 elements in the sample space. The elements of the sample space are as follows:

The circled elements of the sample space are those whose sum is 7.

$$P(7) = \frac{\text{number of successes}}{\text{total number}} = \frac{6}{36} = \frac{1}{6}.$$

The probability, p, of any event is a number such that $0 \le p \le 1$. If $p = 0$, the event cannot happen. If $p = 1$, the event is sure to happen.

EXAMPLE 4

(A) What is the probability of getting a 7 when one die is thrown?

(B) What is the probability of getting a number less than 12 when one die is thrown?

SOLUTIONS

(A) $P(7) = 0$ since a single die has only numbers 1 through 6 on its face.

(B) $P(\# < 12) = 1$ since any face number is less than 12.

The *odds* in favor of an event happening are defined to be the probability of the event happening successfully divided by the probability of the event not happening successfully.

EXAMPLE 5

What are the odds in favor of getting a number greater than 2 when one die is thrown?

$P(\# > 2) = \frac{4}{6} = \frac{2}{3}$ and $P(\# \not> 2) = \frac{2}{6} = \frac{1}{3}$. Therefore, the odds in favor of a number greater than 2 $= \dfrac{P(\# > 2)}{P(\# \not> 2)} = \dfrac{\frac{2}{3}}{\frac{1}{3}} = \dfrac{2}{1}$ or 2:1.

INDEPENDENT EVENTS

Independent events are events that have no effect on one another. Two events are defined to be independent if and only if $P(A \cap B) = P(A) \cdot P(B)$, where $A \cap B$ means both events *A* and *B* happen. If two events are not independent, they are said to be *dependent.*

EXAMPLE 1

If two fair coins are flipped, what is the probability of getting two heads?

Since the flip of each coin has no effect on the outcome of any other coin, these are independent events.

$$P(\text{HH}) = P(\text{H}) \cdot P(\text{H}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

EXAMPLE 2

When two dice are thrown, what is the probability of getting two 5s?

These are independent events because the result of one die does not affect the result of the other.

$$P(\text{two 5s}) = P(5) \cdot P(5) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

EXAMPLE 3

Two dice are thrown. Event *A* is "the sum of 7." Event *B* is "at least one die is a 6." Are A and B independent?

$A = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}$ and
$B = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$.

Therefore, and $P(B) = \frac{11}{36}$. $A \cap B = \{(1,6)(6,1)\}$. Therefore, $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

$P(A) \cdot P(B) = \frac{11}{216} \neq \frac{1}{18}$ Therefore, $P(A \cap B) \neq P(A) \cdot P(B)$, and so events ***A*** and ***B*** are dependent.

EXAMPLE 4

If the probability that John will buy a certain product is $\frac{3}{5}$, that Bill will buy that product is $\frac{2}{3}$, and that Sue will buy that product is $\frac{1}{4}$, what is the probability that at least one of them will buy the product?

Since the purchase by any one of the people does not affect the purchase by anyone else, these events are independent. The best way to approach this problem is to consider the probability that none of them buys the product.

Let A = the event "John does not buy the product."
Let B = the event "Bill does not buy the product."
Let C = the event "Sue does not buy the product."

TIP

To find the probability of "at least one," find 1 – probability of "none."

$$P(A) = 1 - \frac{3}{5} = \frac{2}{5};\quad P(B) = 1 - \frac{2}{3} = \frac{1}{3};\quad P(C) = 1 - \frac{1}{4} = \frac{3}{4}$$

The probability that none of them buys the product $= P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = \frac{2}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{10}$. Therefore, the probability that at least one of them buys the product is .

MUTUALLY EXCLUSIVE EVENTS

In general, the probability of event A happening or event B happening or both happening is equal to the sum of $P(A)$ and $P(B)$ less the probability of both happening. In symbols, $P(A \quad B) = P(A) + P(B) - P(A \cap B)$, where $A \quad B$ means the union of sets A and B. If $P(A \cap B) = 0$, the events are said to be *mutually exclusive.*

EXAMPLE 1

What is the probability of drawing a spade or a king from a deck of 52 cards?

Let A = the event "drawing a spade."
Let B = the event "drawing a king."

Since there are 13 spades and 4 kings in a deck of cards,

$$P(A) = \frac{13}{52} = \frac{1}{4}; \quad P(B) = \frac{4}{52} = \frac{1}{13}$$

$P(A \cap B) = P$ (drawing the king of spades) $= \frac{1}{52}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

These events are *not* mutually exclusive.

TIP

Generally in probability, "and" means multiply and "or" means add.

EXAMPLE 2

In a throw of two dice, what is the probability of getting a sum of 7 or 11?

Let A = the event "throwing a sum of 7."
Let B = the event "throwing a sum of 11."

$P(A \cap B) = 0$, and so these events *are* mutually exclusive.
$P(A \quad B) = P(A) + P(B)$. From the chart in Example 3,

$$P(A) = \frac{6}{36} \text{ and } P(B) = \frac{2}{36}.$$

Therefore,

$$P(A \cup B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}.$$

EXERCISES

1. With the throw of two dice, what is the probability that the sum will be a prime number?

(A) $\frac{4}{11}$

(B) $\frac{7}{18}$

(C) $\frac{5}{12}$

(D) $\frac{5}{11}$

(E) $\frac{1}{2}$

2. If a coin is flipped and one die is thrown, what is the probability of getting a head or a 4?

(A) $\frac{1}{12}$

(B) $\frac{1}{3}$

(C) $\frac{5}{12}$

(D) $\frac{7}{12}$

(E)

3. Three cards are drawn from an ordinary deck of 52 cards. Each card is replaced in the deck before the next card is drawn. What is the probability that at least one of the cards will be a spade?

(A)

(B) $\frac{9}{64}$

(C) $\frac{3}{8}$

(D) $\frac{37}{64}$

(E) $\frac{3}{4}$

4. A coin is tossed three times. Let A = {three heads occur} and B = {at least one head occurs}. What is $P(A \quad B)$?

(A) $\frac{1}{8}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) $\frac{3}{4}$

(E) $\frac{7}{8}$

5. A class has 12 boys and 4 girls. If three students are selected at random from the class, what is the probability that all will be boys?

(A) $\frac{1}{55}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{11}{28}$

(E) $\frac{11}{15}$

6. A red box contains eight items, of which three are defective, and a blue box contains five items, of which two are defective. An item is drawn at random from each box. What is the probability that both items will be nondefective?

(A) $\frac{3}{20}$

(B)

(C) $\frac{5}{13}$

(D) $\frac{8}{13}$

(E) $\frac{17}{20}$

7. A hotel has five single rooms available, for which six men and three women apply. What is the probability that the rooms will be rented to three men and two women?

(A) $\frac{23}{112}$

(B) $\frac{97}{251}$

(C) $\frac{10}{21}$

(D) $\frac{5}{9}$

(E) $\frac{5}{8}$

8. Of all the articles in a box, 80% are satisfactory, while 20% are not. The probability of obtaining exactly five good items out of eight randomly selected articles is

(A) 0.003
(B) 0.013
(C) 0.132
(D) 0.147
(E) 0.800

Answers and Explanations

Probability

1. **(C)** There is 1 way to get a 2, and there are 2 ways to get a 3, 4 ways to get a 5, 6 ways to get a 7, 2 ways to get an 11. Out of 36 elements in the sample space, 15 successes are possible.

2. **(D)** The probability of getting neither a head nor a 4 is $\frac{1}{2}\cdot\frac{5}{6}=\frac{5}{12}$. Therefore, probability of getting either is $1-\frac{5}{12}=\frac{7}{12}$.

3. **(D)** Since the drawn cards are replaced, the draws are independent. The probability that none of the cards was a spade $=\frac{39}{52}\cdot\frac{39}{52}\cdot\frac{39}{52}=\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}=\frac{27}{64}$.
Probability that 1 was a spade

4. **(E)** The only situation when neither of these sets is satisfied occurs when three tails appear. $P(A\cup B)=\frac{7}{8}$.

5. **(D)** There are 16 students altogether. The probability that the first person chosen is a boy is $\frac{12}{16}$. Now there are only 15 students left, of which 11 are boys, so the probability that the second student chosen is also a boy is $\frac{11}{15}$. By the same reasoning, the probability that the third is a boy is $\frac{10}{14}$. Therefore, the probability that the first and the second and the third students chosen are all boys is $\frac{12}{16}\times\frac{11}{15}\times\frac{10}{14}=\frac{11}{28}$.

6. **(B)** Probability of both items being nondefective $=\frac{5}{8}\cdot\frac{3}{5}=\frac{3}{8}$.

7. **(C)** $\binom{6}{3}$ is the number of ways 3 men can be selected. $\binom{3}{2}$ is the number of ways 2 women can be selected. $\binom{9}{5}$ is the total number of ways people can be selected to fill 5 rooms.

$$P(3\text{ men, }2\text{ women})=\frac{\binom{6}{3}\binom{3}{2}}{\binom{9}{5}}=\frac{10}{21}.$$

8. **(D)** Since the problem doesn't say how many articles are in the box, we must assume that it is an unlimited number. The probability of picking 5 satisfactory items (and therefore 3 unsatisfactory ones) is $(0.8)^5(0.2)^3$, and there are $\binom{8}{5}$ ways of doing this. Therefore, the desired probability is $\binom{8}{5}$

$(0.8)^5(0.2)^3 \approx 0.147.$