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EST II MATH LEVEL 1

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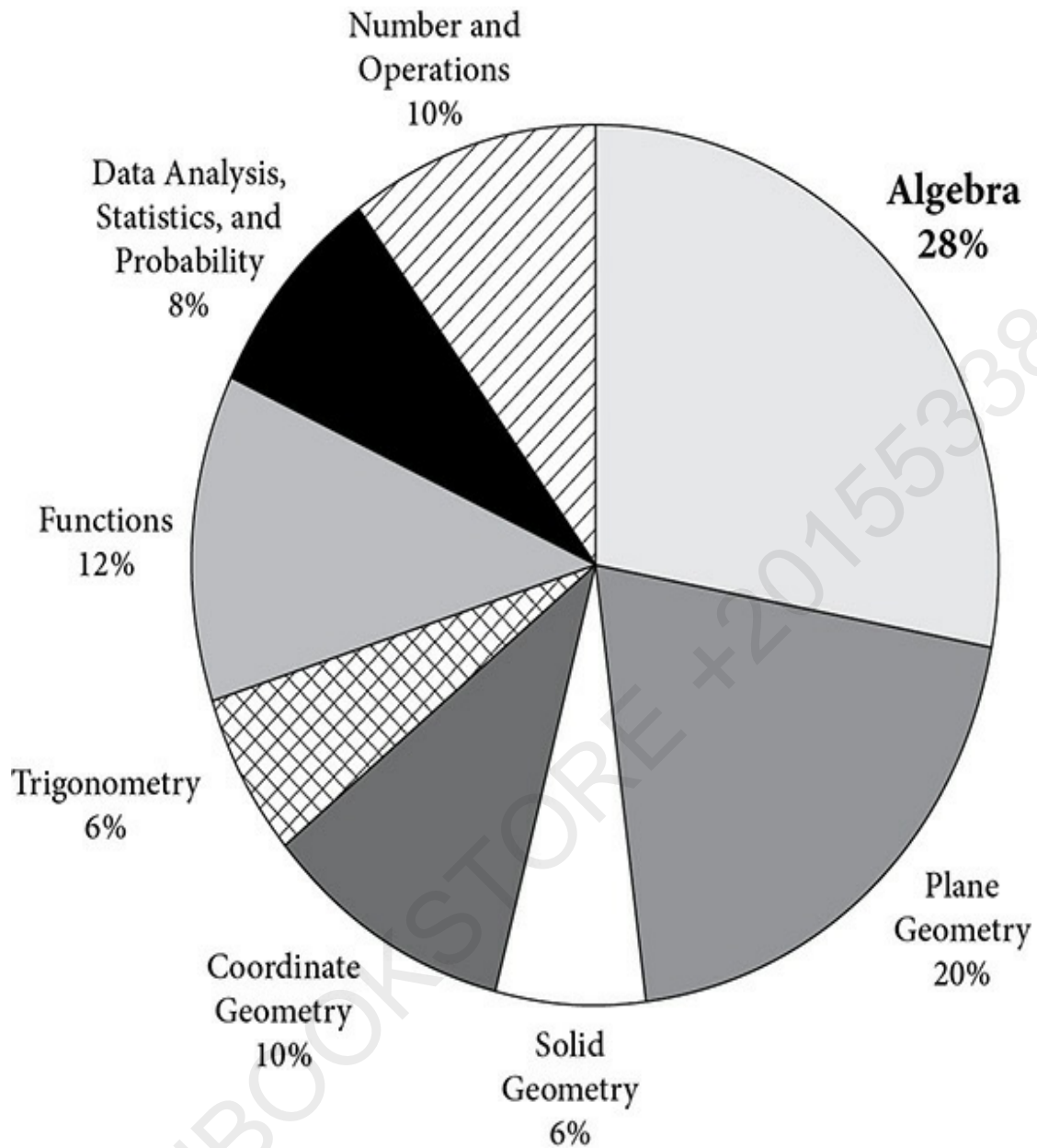
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Algebra

This chapter provides a review of basic algebraic principles. Thirty eight to 42 percent of the Level 1 test questions relate to algebra and functions. That translates to about 28 percent of the test questions relating specifically to algebra. In reality, however, algebra is needed to answer nearly all of the questions on the test, including coordinate geometry, plane geometry, and, especially, functions. The pie chart shows approximately how much of the Level 1 test is related to algebra:



A wide variety of algebra topics are covered in this chapter to help best prepare you for the Math Level 1 test. The topics are:

1. Evaluating Expressions
2. Order of Operations
3. Fractions
 - a. Simplifying Fractions
 - b. Least Common Denominator
 - c. Multiplying Fractions
 - d. Using Mixed Numbers and Improper Fractions

- e. Variables in the Denominator
- 4. Percents
 - a. Converting Percents to Decimals
 - b. Converting Fractions to Percents
 - c. Percent Problems
- 5. Exponents
 - a. Properties of Exponents
 - b. Common Mistakes with Exponents
 - c. Rational Exponents
 - d. Negative Exponents
 - e. Variables in an Exponent
 - f. Logarithms
- 6. Real Numbers
 - a. Vocabulary
 - b. Properties of Real Numbers
 - i. Properties of Addition
 - ii. Properties of Multiplication
 - iii. Distributive Property
 - iv. Properties of Positive and Negative Numbers
- 7. Absolute Value
- 8. Radical Expressions
 - a. Roots of Real Numbers
 - b. Simplest Radical Form
 - c. Rationalizing the Denominator
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- 9. Polynomials
 - a. Vocabulary
 - b. Adding and Subtracting Polynomials
 - c. Multiplying Polynomials
 - d. Factoring

- i. Trinomials
 - ii. Difference of Perfect Squares
 - iii. Sum and Difference of Cubes
- 10. Quadratic Equations
 - a. Factoring
 - b. Quadratic Formula
 - c. Solving by Substitution
 - d. The Discriminant
 - e. Equations with Radicals
- 11. Inequalities
 - a. Transitive Property of Inequality
 - b. Addition and Multiplication Properties
 - c. “And” vs. “Or”
 - d. Inequalities with Absolute Value
- 12. Rational Expressions
 - a. Simplifying Rational Expressions
 - b. Multiplying and Dividing Rational Expressions
 - c. Adding and Subtracting Rational Expressions
 - d. Solving Equations with Rational Expressions
- 13. Systems
 - a. Solving by Substitution
 - b. Solving by Linear Combination
 - c. No Solution vs. Infinite Solutions
 - d. Word Problems with Systems

Evaluating Expressions

Problems asking you to evaluate an expression represent the easiest of the algebra questions on the Math Level 1 test. To answer this type of question, simply substitute the value given for the variable.

Example:

If $x = n^3$ and $n = \frac{1}{2}y$, then find the value of x when $y = 4$.

Substitute $y = 4$ into the second equation to get $n = \frac{1}{2}(4) = 2$.

Now substitute $n = 2$ into the first equation to get $x = 2^3 = 8$.
The correct answer is 8.

Order of Operations

The order of operations specifies which step to do first when solving an equation. Students usually remember the order by “PEMDAS” or “Please Excuse My Dear Aunt Sally.” Both expressions stand for **P**arentheses (or other grouping symbols like brackets, radicals, or fraction bars), **E**xponents, **M**ultiplication/**D**ivision, and **A**ddition/**S**ubtraction.

Within a grouping symbol, work from the innermost to the outermost expression. Evaluate exponents and roots from left to right. Then, perform multiplication and division in order from left to right. Finally, perform addition and subtraction in order from left to right.

Example:

Simplify $16 - 2 + [52 \div (7 - 3)]$.

$$= 16 - 2 + [52 \div 4]$$

$$= 16 - 2 + 13$$

$$= 14 + 13$$

$$= 27 \quad \text{Answer}$$

Fractions

Simplifying Fractions

Fractions are in simplest form when the numerator and denominator have no common factor other than 1. To simplify a fraction, factor both the numerator and denominator.

Example:

Simplify $\frac{4}{12}$.

4 and 12 have a common factor of 4. Divide both the numerator and denominator by 4 to get $\frac{1}{3}$.

Don't cancel terms that are not common factors.

Following is a common mistake:

$$\frac{(x^2 - 4)}{8} \neq \frac{x^2}{2}$$

$x^2 - 4$ and 8 do not have a common factor, so this expression is already in simplest form.

Example:

Simplify $\frac{(3x + 12)}{(3x + 3y)}$

$$= \frac{3(x+4)}{3(x+y)} \quad \text{Factor the numerator and denominator. 3 is a common factor.}$$

$$= \frac{(x+4)}{(x+y)} \quad (x \neq -y) \quad \text{(This restriction is important because you cannot divide by zero!)} \quad \text{Answer}$$

Least Common Denominator

The least common denominator (LCD) of two or more fractions is the least common multiple (LCM) of their denominators. To find the LCD:

1. Factor each denominator completely, and write as the product of prime factors. (Factor trees are usually used for this.)
2. Take the greatest power of each prime factor.
3. Find the product of these factors.

Example:

Find the LCD of $\frac{1}{4}$ and $\frac{7}{30}$.

$$4 = 2 \times 2 = 2^2 \quad 30 = 2 \times 3 \times 5$$

The greatest power of 2 is 2^2 . The greatest power of 3 is 3. The greatest power of 5 is 5.

$$2^2 \times 3 \times 5 = 60$$

60 is the LCD. Answer

The least common denominator is helpful when adding and subtracting fractions.

Example:

Simplify $\frac{1}{4} + \frac{7}{30}$.

You already know that the LCD of 4 and 30 is 60. Rewrite each fraction using 60 as the denominator.

$$\begin{aligned}\frac{1 \times 15}{4 \times 15} &= \frac{7 \times 2}{30 \times 2} \\ &= \frac{15}{60} + \frac{14}{60} \\ &= \frac{29}{60} \quad \text{Answer}\end{aligned}$$

The least common denominator is also used when simplifying complex fractions. A *complex fraction* is a fraction whose numerator or denominator contains one or more fractions. Find the LCD of the simple fractions and multiply the numerator and denominator of the complex fraction by it.

Example:

Simplify $\frac{\frac{5}{x}}{\left(\frac{1}{x} - \frac{1}{5x}\right)}$.

Multiply the numerator and the denominator by the LCD of the simple fractions, $5x$.

$$\begin{aligned}
&= \frac{\frac{5}{x}}{\left(\frac{1}{x} - \frac{1}{5x}\right)} \times \frac{5x}{5x} \\
&= \frac{5 \times 5}{(5 - 1)} \\
&= \frac{25}{4} \quad \text{Answer}
\end{aligned}$$

Multiplying Fractions

To multiply fractions, simply multiply straight across. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Example:

Simplify $\frac{2}{5} \times \frac{3}{6} \times \frac{4}{7}$.

$$= \frac{2 \times 3 \times 4}{5 \times 6 \times 7}$$

$$= \frac{24}{210} \quad \text{Divide by a common factor to simplify.}$$

$$= \frac{4}{35} \quad \text{Answer}$$

You can also simplify the fractions before multiplying to save time.

Example:

$$\begin{aligned} \frac{2}{5} \times \frac{3}{6} \times \frac{4}{7} &= \frac{1}{5} \times \frac{1}{1} \times \frac{4}{7} \\ &= \frac{1 \times 1 \times 4}{5 \times 1 \times 7} \\ &= \frac{4}{35} \end{aligned}$$

Remove the common factors of 2 and 3.

To divide by a fraction, multiply by its reciprocal. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. This is known as the Division Rule for Fractions. Of course, b , c , and d cannot equal zero, since you cannot divide by zero.

Example:

Simplify $18 \div \frac{6}{11}$.

$$= 18 \times \frac{11}{6}$$

$$= 3 \times \frac{11}{1} \quad \text{Divide through by a common factor of 6.}$$

$$= \frac{3 \times 11}{1} = 33 \quad \text{Answer}$$

Using Mixed Numbers and Improper Fractions

A mixed number represents the sum of an integer and a fraction. For instance:

$$3\frac{1}{4} = 3 + \frac{1}{4}$$

In fractional form:

$$3\frac{1}{4} = \frac{4 \times 3 + 1}{4} = \frac{13}{4}$$

When $3\frac{1}{4}$ is written as the fraction $\frac{13}{4}$, it is called an *improper fraction*.

Improper fractions are fractions whose numerator is greater than the denominator. It is often easier to change mixed numerals to improper fractions when simplifying an expression.

Example:

Simplify $8\frac{2}{3} \div \frac{1}{6}$.

$$8\frac{2}{3} = \frac{3 \times 8 + 2}{3} = \frac{24 + 2}{3} = \frac{26}{3} \quad \text{Change to an improper fraction first.}$$

Then $8\frac{2}{3} \div \frac{1}{6} = \frac{26}{3} \times \frac{6}{1}$ Multiply by the reciprocal of $\frac{1}{6}$.

$$\frac{26 \times 6}{3} = \frac{26 \times 2}{1} = 52 \quad \text{Answer}$$

Variables in the Denominator

Fraction problems get more difficult to solve when there is a variable in the denominator. To solve, find the least common denominator (LCD) of the fractions and multiply both sides of the equation by it.

Example:

$$\text{Solve } 4 - \frac{1}{x} = \frac{6}{2x}$$

The LCD of $\frac{1}{x}$ and $\frac{6}{2x}$ is $2x$.

$$2x \left(4 - \frac{1}{x} \right) = 2x \left(\frac{6}{2x} \right)$$

$$8x - 2 = 6$$

$$8x = 8$$

$$x = 1 \quad \text{Answer}$$

Sometimes multiplying both sides of an equation by the LCD transforms the equation into an equation that is NOT equivalent to the original one. Multiplying both sides of an equation by a polynomial may introduce *extraneous roots* that do not satisfy the original equation. It is crucial to go back and check your answer in the original fractional equation.

Example:

$$\text{Solve } \frac{2}{(x^2 - 7x + 10)} = \frac{x - 1}{x - 5}.$$

Factor $x^2 - 7x + 10$ into $(x - 2)(x - 5)$. Now multiply both sides of the equation by the LCD of the fractions, $(x - 2)(x - 5)$.

$$\frac{2}{(x^2 - 7x + 10)} \times (x - 2)(x - 5) = \frac{x - 1}{x - 5} \times (x - 2)(x - 5)$$

$$2 = (x - 1)(x - 2)$$

$$2 = x^2 - 3x + 2$$

$$0 = x^2 - 3x$$

$$0 = x(x - 3)$$

$$x = 0 \text{ or } x = 3 \quad \text{Answer}$$

Since you multiplied both sides of the equation by a polynomial, $(x - 2)(x - 5)$, check to ensure the equation does not have extraneous roots. Substituting $x = 0$ into the original equation results in $\frac{2}{10} = \frac{-1}{-5}$ or $1/5 = 1/5$. Substituting

$x = 3$ into the original equation results in $2/-2 = 2/-2$ or $-1 = -1$. Both answers check, so they are not extraneous.

Proportions are another type of problem that may have variables in the denominator. A *proportion* is an equation that sets two ratios (fractions) equal to each other. Don't worry about finding least common denominators when solving a proportion, simply *cross multiply*.

Example:

$$\frac{10}{x + 4} = \frac{6}{x}$$

$$10x = 6(x + 4)$$

$$10x = 6x + 24$$

$$4x = 24$$

$$x = 6 \quad \text{Answer}$$

Percents

Converting Percents to Decimals

Percent problems can often be easily translated into simpler equations. *Percent* means “per one hundred” or “divided by one hundred.” To convert a percent to a decimal, move the decimal point two places to the left.

$$5\% = 5 \text{ out of } 100 = \frac{5}{100} = 5 \div 100 = 0.05$$

$$\frac{3}{4}\% = \frac{3}{4} \text{ out of } 100 = \frac{\frac{3}{4}}{100} = \frac{3}{4} \div 100 = 0.75 \div 100 = 0.0075$$

The simplest way to change a fractional percent to a decimal is to change the fraction to a decimal first, and then move the decimal point two places to the *left*.

Example:

Simplify $\frac{2}{5}\%$.

$$\frac{2}{5} = 0.4$$

$$0.4\% = 0.004 \quad \text{Answer}$$

Converting Fractions to Percents

The simplest way to change a fraction to a percent is to change the fraction to a decimal first, and then move the decimal point two places to the *right*.

$$\frac{2}{5} = 0.4$$
$$0.4 = 40\%$$

Example:

When written as a percent, $7\frac{1}{4}$ is what value?

$$7\frac{1}{4} = 7.25$$

725% Answer

Percent Problems

In percent problems, the word “of” means “multiply” and the word “is” means “equals.” It is often useful to set up problems in the format ***a is b% of c*** and solve for the unknown variable.

Example:

26 is 25% of what number?

$$26 = 25\% \times c \quad \text{Think } \mathbf{a \text{ is } b\% \text{ of } c.} \quad 26 = a \text{ and } 25 = b.$$

$$26 = \frac{25}{100}(c) \quad \text{so } 26 = 0.25c$$

$$\frac{26}{0.25} = c$$

$$c = 104 \quad \text{Answer}$$

Example:

What percent of 12 is 4? Round answer to the nearest tenth.

$$4 = b\% \times 12 \quad \text{Think } a \text{ is } b\% \text{ of } c. 4 = a \text{ and } 12 = c.$$

$$\frac{4}{12} = b\%$$

$$\frac{4}{12} = \frac{b}{100}$$

$$\frac{4 \times 100}{12} = b$$

$$b = \frac{400}{12} = 33.\bar{3}\% \quad \text{Answer}$$

Example:

Find 85% of 324.

$$a = 85\% \times 324 \quad \text{Think } a \text{ is } b\% \text{ of } c. 85 = b \text{ and } 324 = c.$$

$$a = 0.85 \times 324$$

$$a = 275.4 \quad \text{Answer}$$

Exponents

Properties of Exponents

Given the expression 2^4 , the exponent 4 tells you the number of times the base, 2, is to be used as a factor.

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$

Remember that a number raised to the zero power equals 1.

$$a^0 = 1 \ (a \neq 0) \quad 5^0 = 1$$

Rules of Exponents

1. To multiply two powers with the same base, you *add* the exponents:

$$a^m \times a^n = a^{m+n} \quad 2^2 \times 2^3 = 2^{2+3} = 2^5$$

2. To divide two powers with the same base, you *subtract* the exponents:

$$\frac{a^m}{a^n} = a^{m-n} \quad \frac{2^5}{2^3} = 2^{5-3} = 2^2$$

3. To raise a power to a power, you *multiply* the exponents:

$$(a^m)^n = a^{mn} \quad (2^2)^3 = 2^{2 \times 3} = 2^6$$

4. To raise a product to a power, you *raise each factor to the power* and *multiply*:

$$(ab)^m = a^m \times b^m \quad (2 \times 5)^3 = 2^3 \times 5^3$$

5. To raise a quotient to a power, you *raise each factor to the power* and *divide*:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \left(\frac{8}{4}\right)^2 = \frac{8^2}{4^2}$$

When simplifying expressions involving the rules of exponents, it is easiest to simplify each variable separately as shown in the next example.

Example:

Simplify $\frac{30x^3y^5}{-5x^2y^3}$. (Assume x and y do not equal 0.)

$$\frac{30x^3y^5}{-5x^2y^3} = \frac{30}{-5} \times \frac{x^3}{x^2} \times \frac{y^5}{y^3} \quad \text{Isolate each variable.}$$

$$= -6 \times x^{3-2} \times y^{5-3} \quad \text{Use rule \#2. Subtract the exponents for each base.}$$

$$= -6 \times x^1 \times y^2$$

$$= -6xy^2 \quad \text{Answer}$$

Example:

Simplify $\frac{(a^3bc^3)^2}{(a^2bc)^2}$. (Assume a , b , and c do not equal 0.)

$$\frac{(a^3bc^3)^2}{(a^2bc)^2} = \frac{a^{3 \times 2} b^2 c^{3 \times 2}}{a^{2 \times 2} b^2 c^2} \quad \text{Use rules \#3 and \#5. Multiply the exponents.}$$

$$= \frac{a^6}{a^4} \times \frac{b^2}{b^2} \times \frac{c^6}{c^2} \quad \text{Isolate each variable.}$$

$$= a^2 \times b^0 \times c^4 \quad \text{Use rule \#2. Subtract the exponents for each base.}$$

$$= a^2 \times 1 \times c^4$$

$$= a^2c^4 \quad \text{Answer}$$

Common Mistakes with Exponents

When studying for the Subject Test, make sure you don't make these common mistakes:

1. $\left(-\frac{1}{2}\right)^{-3} \neq \frac{1}{8}$

Watch your negative exponents. The quantity $\frac{1}{2}$ should be put in the denominator and raised to the third power:

$$\left(-\frac{1}{2}\right)^{-3} = \frac{1}{\left(-\frac{1}{2}\right)^3} = \frac{1}{\left(\frac{(-1)^3}{2^3}\right)} = \frac{8}{-1} = -8$$

2. $3x^3 \neq 27x^3$

$$3x^3 \neq (3x)^3$$

Only x is raised to the third power here. $3x^3$ is in simplest form.

3. $2^2 + 2^3 \neq 2^5$

You only add the exponents when finding the *product* of two terms in the same base. $2^2 \times 2^3$ would in fact equal 2^5 . There's no rule of exponents that applies to finding the *sum* of two terms, so just simplify each term.

$$2^2 + 2^3 = 4 + 8 = 12$$

4. $4^3 \times 4^4 \neq 4^{12}$

Remember to add the exponents when the bases are the same.

$$4^3 \times 4^4 = 4^7$$

5. $2^2 \times 2^3 \neq 4^5$

It's correct to add the exponents here, but the base should remain unchanged.

6. $(a + b)^2 \neq a^2 + b^2$

This is a very common error. It is important to understand that raising the quantity $a + b$ to the second power means that the base, $a + b$, is to be used as a factor two times. In other words, $(a + b)^2 = (a + b)(a + b)$. You then need to multiply using the FOIL method:

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

Rational Exponents

The previous examples focus on integral exponents. It is possible, however, to define a^x when x is any *rational number*. Remember that a rational number

can be expressed as $\frac{p}{q}$ and results from dividing an integer by another

(nonzero) integer.

$4^{\frac{1}{2}}$ reads as "4 to the one-half power" and equals $\sqrt{4}$.

$5^{\frac{2}{3}}$ reads as "5 to the two-thirds power" and equals $\sqrt[3]{5^2}$ or $\sqrt[3]{25}$. $5^{\frac{2}{3}}$ can

also be represented as $(\sqrt[3]{5})^2$.

$x^{\frac{a}{b}}$ equals $\sqrt[b]{x^a}$ or $(\sqrt[b]{x})^a$ as long as $b \neq 0$. All of the rules of exponents previously discussed also apply to rational exponents.

Example:

Simplify $81^{\frac{3}{4}}$.

$$\begin{aligned}81^{\frac{3}{4}} &= (\sqrt[4]{81})^3 \\ &= 3^3 = 27 \quad \text{Answer}\end{aligned}$$

This problem can also be solved by raising 81 to the third power first.

$$\sqrt[4]{81^3} = \sqrt[4]{531,441}$$

Finding the fourth root of 531,441 is less obvious than finding the fourth root of 81 as in the solution above.

Example:

Simplify $(8^4)^{\frac{1}{12}}$.

$$\begin{aligned}(8^4)^{\frac{1}{12}} &= 8^{\frac{4}{12}} = 8^{\frac{1}{3}} \\ &= (2^3)^{\frac{1}{3}} = 2^1 = 2 \quad \text{Answer}\end{aligned}$$

Negative Exponents

Given the expression 2^{-4} , rewrite it without the negative exponent by moving 2^4 to the denominator. In other words, 2^{-4} is the reciprocal of 2^4 .

$$2^{-4} = \frac{1}{2^4} = \frac{1}{(2 \times 2 \times 2 \times 2)} = \frac{1}{16}$$

Remember that expressions in simplest form typically do not contain negative exponents.

Example:

Simplify $\frac{x^{-3}}{2y^3} \left(\frac{1}{xy} \right)^{-3}$. (Assume x and y do not equal 0.)

$$\frac{x^{-3}}{2y^3} \left(\frac{1}{xy} \right)^{-3} = \frac{x^{-3}}{2y^3} \left(\frac{1^{-3}}{x^{-3}y^{-3}} \right) \quad \text{Use rule \#5.}$$

$$= \left(\frac{1}{2x^3y^3} \right) \left(\frac{x^3y^3}{1} \right) \quad \text{Simplify the negative exponents.}$$

$$= \frac{x^3y^3}{2x^3y^3}$$

$$= \frac{1}{2} \quad \text{Answer}$$

Variables in an Exponent

Solving an equation with a variable in the exponent can be easily done if both sides can be rewritten in the same base.

Example:

Solve $4^x = 32^{x+1}$. Recognize that 4 and 32 can be written in base 2.

$$(2^2)^x = (2^5)^{x+1}$$

$2^{2x} = 2^{5(x+1)}$ Since both sides are in base 2, set the exponents equal and solve.

$$2x = 5(x + 1)$$

$$2x = 5x + 5$$

$$-3x = 5$$

$$x = -\frac{5}{3} \quad \text{Answer}$$

Logarithms

A *logarithm* is the inverse of *exponentiation*. It represents the power to which the base must be raised to achieve a certain number.

In $3^2 = 9$, 3 is the *base*, the raised 2 is the *exponent*, and 9 is the result when 3 is raised to the second power. The *logarithmic expression* is $\log_3 9 = 2$ because 2 is the power 3 must be raised in order to produce the result 9.

Example:

Solve $\log_4 64$.

This means “4 raised to what power = 64?” and can be rewritten in exponential form as $4^x = 64$.

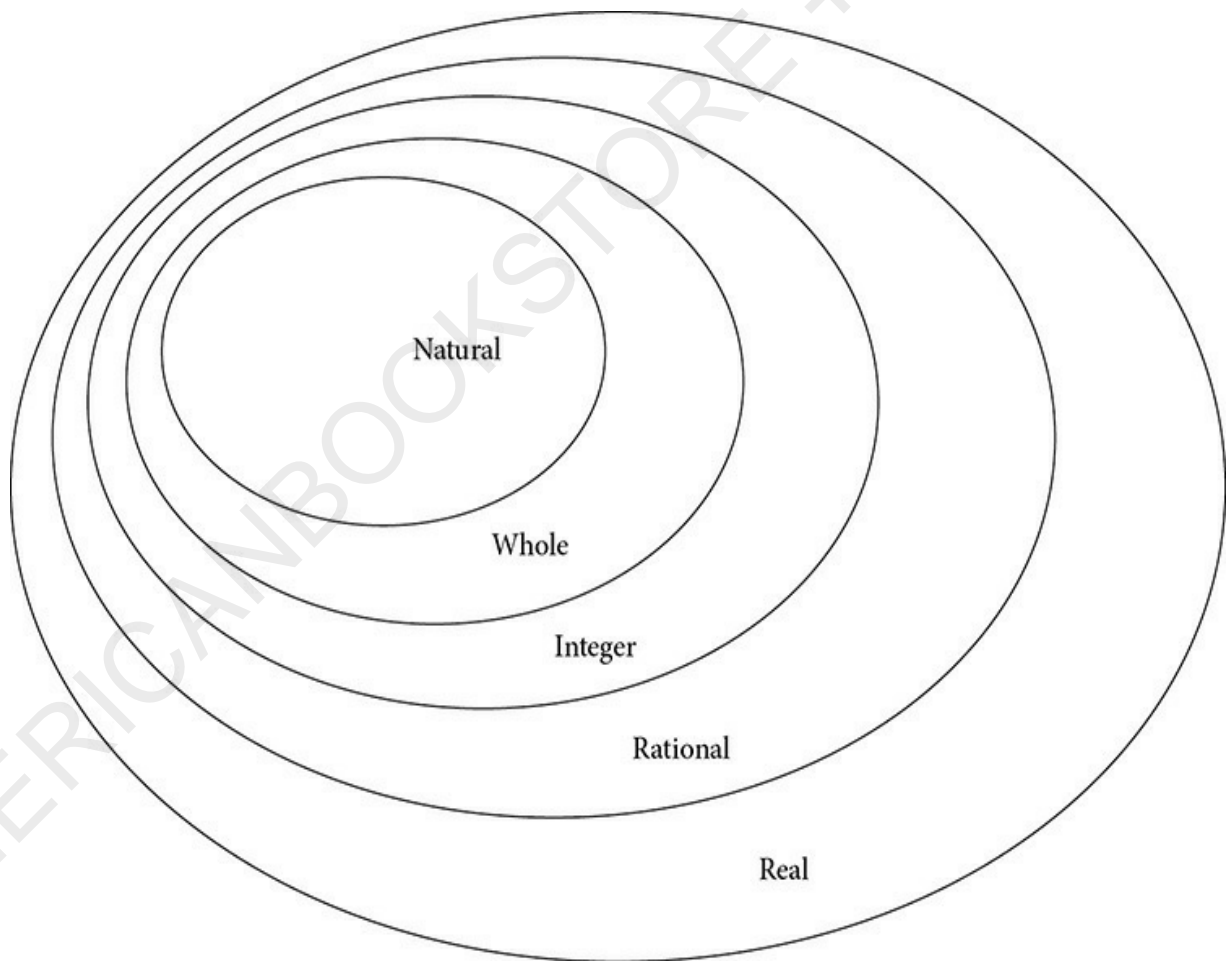
$$4^3 = 64$$

$$\log_4 64 = 3 \quad \text{Answer}$$

Real Numbers

Vocabulary

Natural Numbers	$\{1, 2, 3, \dots\}$
Whole Numbers	$\{0, 1, 2, 3, \dots\}$
Integers	$\{\dots -2, -1, 0, 1, 2, 3, \dots\}$
Rational Numbers	Numbers that can be expressed as $\frac{p}{q}$ and result from dividing an integer by another (nonzero) integer. $-\frac{2}{3}, \frac{3}{4}, 0.7, 1.333, \dots$ and 8 are examples of rational numbers.
Irrational Numbers	Numbers that cannot be expressed as $\frac{p}{q}$. In decimal form, they are nonterminating and nonrepeating. π and $\sqrt{2}$ are examples of irrational numbers.
Real Numbers	The set of all rational and irrational numbers.



Example:

Which of the following is NOT an irrational number?

- (A) $\sqrt{2}$ (B) π (C) $\sqrt{50}$ (D) 1.33333... (E) 5.020020002...

1.33333 is a repeating decimal so it is rational. Irrational numbers are infinite and nonrepeating, such as answer E. 5.02020202... is a repeating decimal, but 5.020020002... is nonrepeating, since the number of zeros continues to increase.

(D) 1.333333 Answer

Properties of Real Numbers

It is important to understand the properties of real numbers in order to move on in algebra. Before getting into the operations of addition and multiplication, let's review examples of the basic properties of equality. These hold true for all real numbers.

- **Reflexive Property** $x = x$ $2 = 2$
- **Symmetric Property** If $a = b$, then $b = a$. If $x = 1$, then $1 = x$.
- **Transitive Property** If $a = b$ and $b = c$, then $a = c$. If $x = 1$ and $1 = y$, then $x = y$.
- **Addition Property** If $a = b$, then $a + c = b + c$. If $x = 4$, then $x + 2 = 4 + 2$.
- **Multiplication Property** If $a = b$, then $ac = bc$. If $x = 9$, then $3x = 3 \times 9$.

The last two properties show the fundamental principle of algebra:

What you do to one side of an equation, you MUST do to the other side.

In other words, if you add 2 to one side of an equation, you must add 2 to the other side to maintain the equality. You can perform nearly any operation on one side as long as you also perform it on the other. Most of this algebra chapter comes back to this fundamental idea. If you take the square root of one side of an equation, you must take the square root of the other side. If you cube one side of an equation, you must cube the other. You get the idea.

As you review the following properties of addition and multiplication, remember these are the properties that allow you to perform operations needed to simplify expressions and solve equations. It is important to be able to perform these operations *and* identify what property is being used. Assume a , b , and c are real numbers.

Properties of Addition

1. Closure Property of Addition

The sum $a + b$ results in a unique real number.

$3 + 11$ equals the real number 14.

2. Commutative Property of Addition

$$a + b = b + a$$

$$20 + 5 = 5 + 20$$

3. Associative Property of Addition

$$(a + b) + c = a + (b + c)$$

$$(8 + 9) + 10 = 8 + (9 + 10)$$

4. Identity Property of Addition (Additive Identity)

There is a unique real number *zero* such that:

$$a + 0 = a$$

$$77 + 0 = 77$$

5. Property of Opposites (Additive Inverse)

For each real number a , there is a real unique number $-a$ such that:

$$a + -a = 0$$

$-a$ is the *opposite* of a . It is also called the *additive inverse* of a .

$$6 + -6 = 0$$

Properties of Multiplication

1. Closure Property of Multiplication

The product $a \times b$ results in a unique real number.

3×11 equals the real number 33.

2. Commutative Property of Multiplication

$$ab = ba$$

$$20 \times 5 = 5 \times 20$$

3. Associative Property of Multiplication

$$(a \times b) \times c = a \times (b \times c)$$

$$(8 \times 9) \times 10 = 8 \times (9 \times 10)$$

4. Identity Property of Multiplication (Multiplicative Identity)

There is a unique real number *one* such that:

$$a \times 1 = a$$

$$77 \times 1 = 77$$

5. Property of Reciprocals (Multiplicative Inverse)

For each real number a (except 0), there is a unique real number $\frac{1}{a}$ such

that:

$$a \times \frac{1}{a} = 1$$

$\frac{1}{a}$ is the *reciprocal* of a . It is also called the *multiplicative inverse* of a .

Zero has no reciprocal, since $\frac{1}{0}$ is undefined.

$$6 \times \frac{1}{6} = 1$$

Distributive Property

The distributive property of multiplication over addition states that:

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

$$3(5 + 6) = (3 \times 5) + (6 \times 3) = 15 + 18 = 33$$

Example:

Simplify $2(6 - x + 2y)$.

$$2(6 - x + 2y) = (2 \times 6) - (2 \times x) + (2 \times 2y) \quad \text{Distribute the 2.}$$

$$= 12 - 2x + 4y \quad \text{Answer}$$

The example shows that the distributive property also works for trinomials (expressions containing three terms). Does it apply to multiplication over subtraction? Try the next example to see that it does hold true.

Example:

Simplify $5(4x^2 - 10)$.

$$5(4x^2 - 10) = (5 \times 4x^2) - (5 \times 10) \quad \text{Distribute the 5.}$$

$$= 20x^2 - 50 \quad \text{Answer}$$

Properties of Positive and Negative Numbers

You should be familiar with working with signed numbers by this point in your math career. Here is a review of the basic properties of positive and negative numbers.

- A positive number times a positive number equals a positive number.
 $3 \times 4 = 12$
- A positive number times a negative number equals a negative number.
 $3 \times -4 = -12$
- A negative number times a negative number equals a positive number.
 $-3 \times -4 = 12$
- To subtract a number, add its opposite.
 $3 - 4 = 3 + (-4) = -1$
 $3 - (-4) = 3 + (+4) = 7$
- The sum of the opposites of two numbers is the opposite of their sum.
 $(-3) + (-4) = -(3 + 4) = -7$
- $x(-1) = -x$ for all real values of x .

$$3(-1) = -3 \text{ and } -3(-1) = -(-3) = 3$$

- $x(-y) = -(xy)$ for all real values of x and y .

$$3 \times (-4) = -(3 \times 4) = -12$$

- $x - y = -(y - x)$

This is a variation of the property of opposites that usually tricks students. It is often used when simplifying rational expressions and factoring, so it is important to recognize.

$$\frac{(x - y)}{(y - x)} = -1 \quad \frac{(x - 4)}{(4 - x)} = -1 \quad (x - 2)(2 - x) = -(x - 2)^2$$

Absolute Value

The absolute value of a number is the distance from the graph of the number on a number line to the origin. For a real number x , the absolute value of x is defined as:

$$|x| = x, \text{ if } x > 0$$

$$|x| = 0, \text{ if } x = 0$$

$$|x| = -x, \text{ if } x < 0$$

Example:

Evaluate the expression $|x| - 2|y|$ if $x = 6$ and $y = -3$.

$$|x| - 2|y| = |6| - 2|-3| \quad \text{Substitute the given values.}$$

$$= 6 - 2 \times 3$$

$$= 6 - 6 = 0 \quad \text{Answer}$$

To solve an equation involving absolute value, think in terms of two separate equations: one where the expression inside the absolute value signs is *positive* and one where the expression inside the absolute value is *negative*.

Example:

Solve $|x - 3| = 1$.

$x - 3 = 1$ or $-(x - 3) = 1$ Recognize when the expression $x - 3$ is positive and negative.

$x = 4$ or $-x = -2$

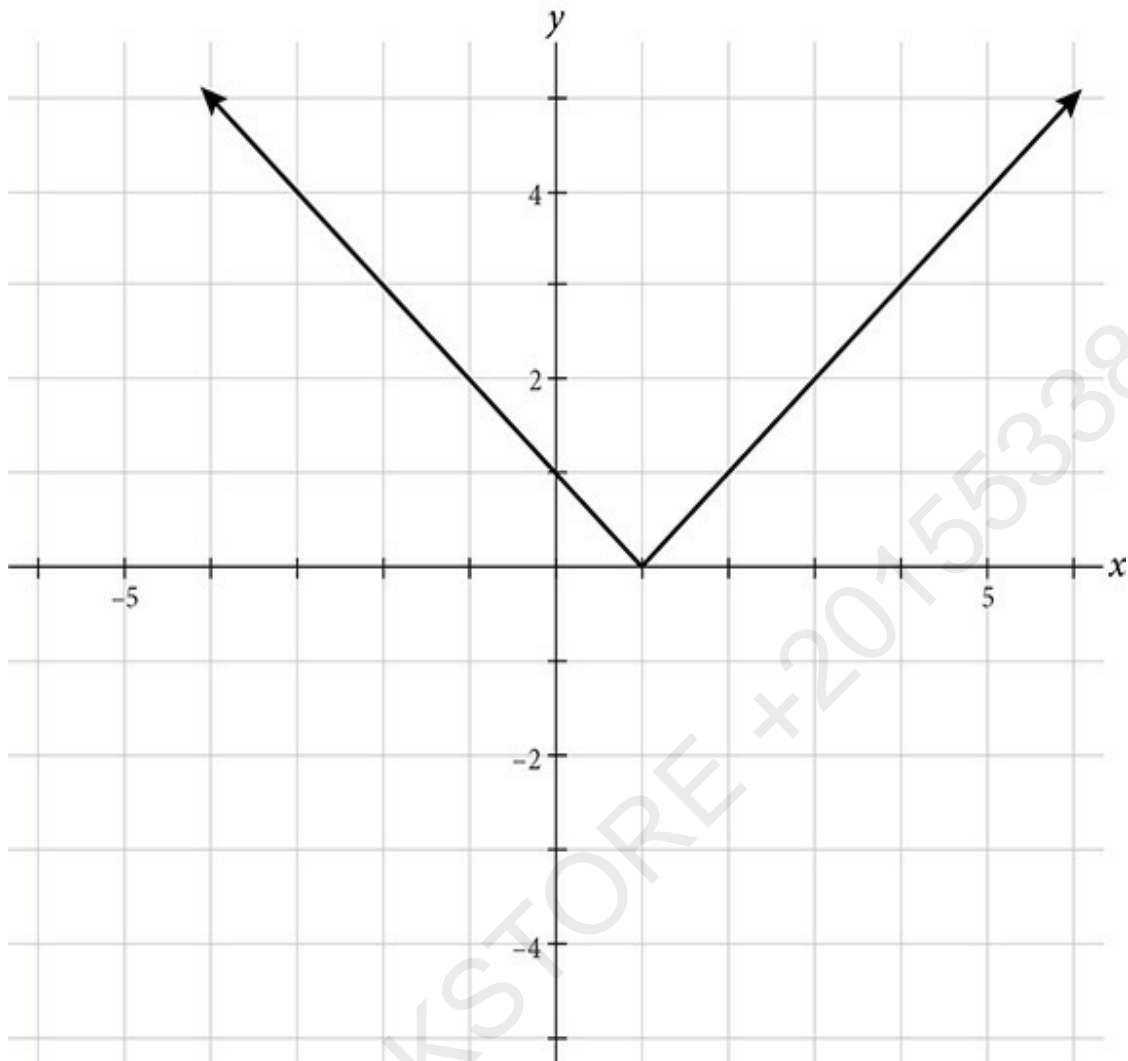
The solution set is $\{2, 4\}$. Answer

Example:

Graph $y = |x - 1|$.

It is important to recognize immediately that y will never be negative, since it equals the absolute value of an expression. Absolute value, by definition, is a positive distance. If you're unsure what happens to an absolute value graph, try plotting a few points to see what results.

When $x = 1$, $y = |1 - 1| = 0$, so the coordinate $(1,0)$ must be part of the solution. When $x = 2$, $y = |2 - 1| = |1| = 1$, so the coordinate $(2,1)$ must be part of the solution. When $x = 0$, $y = |0 - 1| = |-1| = 1$, so the coordinate $(0,1)$ must be part of the solution. These three points are enough to sketch the graph, which resembles a letter "V" and has a vertex of $(1,0)$.



Radical Expressions

Roots of Real Numbers

A *radical* is a symbol such as $\sqrt[n]{x}$ where n is the *index* and x is the *radicand*. If n is not written, it is assumed to equal 2, for example, \sqrt{x} (read the *square root* of 4). Every positive number x has two square roots: \sqrt{x} and $-\sqrt{x}$. The square roots of 16 are 4 and -4 because both 4^2 and $(-4)^2$ equal 16. The positive square root, 4 in this case, is called the *principal square root*.

Example 1:

Solve $x^2 = 49$.

$x = 7$ or -7

The roots are 7 and -7 .

Two solutions

Example 2:

Solve $\sqrt{x} = 6$.

Square both sides to get $x = 6^2$.

$x = 36$

One solution

It is important to note the distinction in the examples above. When finding the roots of an equation where the variable is raised to an *even* power, remember there's a positive and negative solution. In Example 1, it is true that 49 has two real roots, but $\sqrt{49}$ has only one solution. $\sqrt{49} \neq -7$. If the variable is raised to an odd power, however, there's only *one real root*. For instance, there is only one solution for x in the following example:

Example:

Solve $x^3 = -125$.

Take the cube root of both sides to get $x = \sqrt[3]{-125}$.

$x = -5$ Answer

Check your answer by substituting it back into the original equation. (This is always a good idea!)

$$(-5)^3 = (-5)(-5)(-5) = -125$$

When working with radicals, remember the product and quotient properties:

- **The Product Property of Radicals**

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$$

$$\sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$$

- **The Quotient Property of Radicals**

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[3]{\frac{64}{8}} = \frac{\sqrt[3]{64}}{\sqrt[3]{8}} = \frac{4}{2} = 2$$

Be careful not to apply the product and quotient properties to finding the sum of radicals.

$$\sqrt{3^2 + 4^2} = \sqrt{5^2} \text{ but } \sqrt{3^2} + \sqrt{4^2} \neq \sqrt{5^2}$$

Simplest Radical Form

Make sure two things are true when writing an expression, such as $\sqrt[n]{x}$, in *simplest radical form*:

1. Factor all perfect n th powers from the radicand.
2. Rationalize the denominators so that no radicals remain in the denominator and no radicands are fractions.

Write the following examples in simplest radical form.

Example 1:

$$\begin{aligned}\sqrt{75} \\ = \sqrt{25 \times 3}\end{aligned}$$

25 is the largest perfect square factor of 75. 27 is the largest perfect cube factor of 54.

$$\begin{aligned}= \sqrt{25} \times \sqrt{3} \\ = 5\sqrt{3}\end{aligned}$$

Example 2:

$$\begin{aligned}\sqrt[3]{54} \\ = \sqrt[3]{27 \times 2}\end{aligned}$$

$$\begin{aligned}= \sqrt[3]{27} \times \sqrt[3]{2} \\ = 3\sqrt[3]{2}\end{aligned}$$

Rationalizing the Denominator

To rationalize a denominator containing a radical, try to create a perfect square, cube, or other n th power. Let's look at a square root with a fractional radicand first.

Example:

Simplify $\sqrt{\frac{7}{3}}$.

$$= \sqrt{\frac{7}{3} \times \frac{3}{3}}$$

Multiplying the denominator by 3 will create a perfect square, 3^2 , in the denominator.

$\frac{3}{3}$ is another way of writing **1**, the identity element for multiplication.

$$= \sqrt{\frac{21}{9}}$$

$$= \sqrt{\frac{21}{3}}$$

Answer

Solving a cube root with a fractional radicand is nearly identical. You must multiply by a perfect square this time in order to create a perfect cube in the denominator.

Example:

Simplify $\sqrt[3]{\frac{15}{2}}$.

$$= \sqrt[3]{\frac{15}{2} \times \frac{2^2}{2^2}}$$

Multiplying the denominator by 2^2 will create a perfect cube, 2^3 , in the denominator.

$\frac{2^2}{2^2}$ is another way of writing **1**, the identity element for multiplication.

$$= \sqrt[3]{\frac{60}{8}}$$

$$= \sqrt[3]{\frac{60}{2}}$$

Answer

Now let's look at rationalizing a denominator that contains a radical.

Example:

Simplify $\frac{(\sqrt{20} - \sqrt{6})}{\sqrt{3}}$.

$$= \left(\frac{\sqrt{20} - \sqrt{6}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \right)$$

Multiplying the denominator by $\sqrt{3}$ will create a perfect square in the denominator's radicand.

$\frac{\sqrt{3}}{\sqrt{3}}$ is another way of writing **1**.

$$= \frac{[\sqrt{3}(\sqrt{20} - \sqrt{6})]}{\sqrt{9}}$$

Distribute $\sqrt{3}$.

$$= \frac{(\sqrt{60} - \sqrt{18})}{3}$$

The denominator is rationalized here, but the numerator is not in simplest radical form.

$$= \frac{(2\sqrt{15} - 3\sqrt{2})}{3}$$

Answer

Conjugates

Conjugates are expressions in the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Assuming a and b are integers even if a or b is a perfect square, the product of conjugates will always equal an integer.

$$(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = \sqrt{a}\sqrt{a} - \sqrt{a}\sqrt{b} + \sqrt{b}\sqrt{a} - \sqrt{b}\sqrt{b} = a - b$$

$$(\sqrt{19} + \sqrt{3}) \times (\sqrt{19} - \sqrt{3}) = \sqrt{19}\sqrt{19} - \sqrt{19}\sqrt{3} + \sqrt{3}\sqrt{19} - \sqrt{3}\sqrt{3} = 19 - 3 = 16$$

Conjugates are useful when rationalizing a denominator containing a *binomial radical expression*.

Example:

Simplify $\frac{(1 + \sqrt{2})}{(1 - \sqrt{2})}$.

$$\frac{(1 + \sqrt{2})}{(1 - \sqrt{2})} \times \frac{(1 + \sqrt{2})}{(1 + \sqrt{2})}$$

Multiply both numerator and denominator by the conjugate of the denominator $1 + \sqrt{2}$.

$\frac{(1 + \sqrt{2})}{(1 + \sqrt{2})}$ is another way of writing **1**.

$$= \frac{(1 + 2\sqrt{2} + 2)}{1 - 2}$$

$$= \frac{(3 + 2\sqrt{2})}{-1}$$

The denominator becomes an integer, -1 . Now simplify.

$$-3 - 2\sqrt{2} \quad \text{Answer}$$

Polynomials

Vocabulary

A *monomial* is a single term, such as a constant, a variable, or the product of constants and variables. **7**, x , $5x^2$, and $-3xy^2$ are examples of monomials. A polynomial contains many terms. By definition, a *polynomial* is the sum of monomials. $x^2 - 3x + 2$ is an example of a polynomial containing the *terms* x^2 , $3x$, and 2 .

Polynomials can be added, subtracted, multiplied, and divided following the properties of real numbers.

Adding and Subtracting Polynomials

Polynomials can be added and subtracted by *combining like terms*. *Like terms* have the same variables raised to the same power. In other words, they're

terms that differ only by their coefficients.

a^2b^3 and $-4a^2b^3$ are like terms that can be combined. $a^2b^3 + (-4a^2b^3) = -3a^2b^3$. x^2y and $2xy$ are not like terms.

Example:

Subtract $x^2 - 2x + 5$ from $3x - 4$.

$$3x - 4 - (x^2 - 2x + 5)$$

Set up your expression. Subtract the first term from the second.

$$= 3x - 4 - x^2 + 2x - 5$$

Change the subtraction to *adding the opposites*.

$$= -x^2 + (3 + 2)x + (-4 + -5)$$

Combine like terms.

$$= -x^2 + 5x - 9 \quad \text{Answer}$$

Multiplying Polynomials

Multiplying polynomials involves using the distributive property. You are probably most familiar with multiplying a binomial (a polynomial with two terms) by another binomial. For example:

$$(x + 4)(x - 6)$$

The *FOIL method* helps to remember how to multiply terms: **F**irst, **O**uter, **I**nner, **L**ast.

$$(x + 4)(x - 6) = x^2 - 6x + 4x - 24$$

Now, add the outer and inner products to simplify the expression.

$$= x^2 - 2x - 24$$

Example:

Multiply $(2x + 7)(2x - 7)$.

$$(2x + 7)(2x - 7) = 4x^2 - 14x + 14x - 49$$

$$= 4x^2 + (-14x + 14x) - 49$$

$$= 4x^2 - 49 \quad \text{Answer}$$

FOIL

Notice that the inner and outer terms are opposites.

Example:

$$(5a + 1)(1 - 5a) = (5a + 1)(-(5a - 1))$$

Rewrite $(1 - 5a)$ as the opposite of $(5a - 1)$.

$$= -(25a^2 - 5a + 5a - 1) \quad \text{FOIL}$$

$$= -(25a^2 + 5a - 5a + 1) \quad \text{Distribute the } -1.$$

$$= -25a^2 + 1 \quad \text{Answer}$$

Remembering these special products of polynomials will help when factoring.

SPECIAL PRODUCTS	EXAMPLES
$(a + b)^2 = a^2 + 2ab + b^2$	$(2x + 9)^2 = 4x^2 + 36x + 81$
$(a - b)^2 = a^2 - 2ab + b^2$	$(x - 12)^2 = x^2 - 24x + 144$
$(a + b)(a - b) = a^2 - b^2$	$(x + 3y)(x - 3y) = x^2 - 9y^2$

Factoring

Now that you've reviewed multiplying polynomials, let's look at its reverse: finding factors. *Factoring* means expressing a polynomial as a product of other polynomials. The most basic way to factor a polynomial is to distribute out its *greatest monomial factor*. (Remember a monomial is a single term, such as a constant, a variable, or the product of constants and variables.)

Just as 12 can be factored as $2^2 \times 3$, the polynomial $12xy^3 - 2x^2y^2$ can be factored as $2xy^2(6y - x)$. $2xy^2$ is the *greatest monomial factor*, otherwise known as their *GCF*, of the terms $12xy^3$ and $2x^2y^2$.

Example:

Factor $xy^3 - xy^2 + x^2y$.

$xy^3 - xy^2 + x^2y = xy(y^2 - y + x)$ The GCF of the terms is xy .

Trinomials

Some trinomials can be factored by recognizing that they're a special product. These are called *perfect square trinomials*:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Example:

Factor $x^2 - 18x + 81$.

$x^2 - 18x + 81 = (x - 9)(x - 9)$ Recognize that half of the second term squared equals the third term. ($\frac{1}{2} \times 18 = 9$ and $9^2 = 81$)
 $= (x - 9)^2$

When you are factoring, it is a good idea to check your work by multiplying the factors to get the original polynomial. Using the FOIL method, you can quickly check that $(x - 9)(x - 9)$ does, in fact, equal $x^2 - 18x + 81$.

Example:

Factor $x^2 + x + \frac{1}{4}$.

$$x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)\left(x + \frac{1}{2}\right) \quad \text{Recognize that } \frac{1}{2}(1) = \frac{1}{2} \text{ and } \frac{1}{2^2} = \frac{1}{4}.$$
$$x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$$

Of course, not every trinomial is a perfect square trinomial that can be expressed as a special product. For these problems, try thinking about the FOIL method and use trial and error.

$$x^2 + 13x + 22 = (? + ?)(? + ?)$$

First, $x \times x = x^2$, so begin the factorization with $(x + ?)(x + ?)$.

Last, $22 = 1 \times 22$ or 2×11 . Think about the factors of the last term. What two numbers will multiply to give you 22 and add to give you 13? (13 is the sum of the **O**uter and **I**nnner terms.) 2 and 11 work: $(x + 2)(x + 11)$.

You're not done yet. Multiply the binomials to check your work. Since the three terms in the original trinomial are positive, both binomials contain positive terms. $(x + 2)(x + 11) = x^2 + 11x + 2x + 22 = x^2 + 13x + 22$. $(x + 2)(x + 11)$ is the correct answer.

Example:

Factor $x^2 - 2x - 35$.

$$x^2 - 2x - 35 = (x + ?)(x + ?) \quad \text{Think about the First terms: } x \times x = x^2.$$

$$35 = 1 \times 35 \text{ or } 5 \times 7$$

What two numbers will multiply to give you -35 and add to give you -2 ? 5 and -7 work.

$$= (x + 5)(x - 7)$$

Check your work. Pay special attention to the positive and negative signs.

Difference of Perfect Squares

The product of the sum of two terms and the difference of the same terms is called the *difference of perfect squares*. When factoring, it is important to recognize that this is a special product.

$$a^2 - b^2 = (a + b)(a - b)$$
$$x^2 - 1 = x^2 - 1^2 = (x + 1)(x - 1)$$

Example:

Factor $4x^2 - 9$.

$$4x^2 - 9 = (2x + 3)(2x - 3) \quad \text{Answer Since } 4x^2 - 9 = (2x)^2 - 3^2.$$

Since both $4x^2$ and 9 are perfect squares, this binomial is a special product. Remembering this will save you time when factoring.

Example:

Factor $x^3 - 64x$.

$$x^3 - 64x = x(x^2 - 64) \quad \text{Always factor out the greatest common factor first.}$$

Here the GCF is x . x^2 and 64 are perfect squares.

$$= x(x + 8)(x - 8) \quad \text{Answer}$$

Note that there is not a rule called “Sums of Perfect Squares.” An expression such as $x^2 + 4$ cannot be further factored.

Sum and Difference of Cubes

Factoring the sum and difference of cubes is not obvious. , be able to recognize these equations:

1. Sum of Cubes

$$(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$$

2. Difference of Cubes

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$$

Example:

Factor $8x^3 - 125$.

Using the equation for the difference of cubes, substitute $a = 2x$ and $b = 5$.

$$8x^3 - 125 = (2x - 5)(4x^2 + 10x + 25) \quad \text{Answer} \quad \text{Since } 8x^3 = (2x)^3 \text{ and } 125 = 5^3.$$

Memorizing the perfect cubes ($1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, etc.) will help you when factoring the sum and difference of cubes.

Quadratic Equations

Factoring

Up to this point, when given a problem such as $x^2 - 8x + 9$, you would say the polynomial is unfactorable. In other words, it's prime. $x^2 - 8x + 9$ doesn't fit any of the three special products, and factoring the trinomial also doesn't work.

$$x^2 - 8x + 9 = (x + ?)(x + ?)$$

What two numbers will multiply to give you 9 and add to give you -8 ? At first glance, 9 and 1 seem to work resulting in four possibilities:

$$(x + 9)(x + 1) \quad (x - 9)(x - 1) \quad (x + 9)(x - 1) \quad (x - 9)(x + 1)$$

None of the four, however, give you a positive 9 constant and a negative 8 coefficient for x . The *Quadratic Formula* can be used to solve a trinomial equation (in the form of a quadratic equation) whether it's factorable or not.

Before discussing the Quadratic Formula, let's take a look at quadratic equations in general. Polynomials in the form $ax^2 + bx + c$ ($a \neq 0$) are quadratic. (*Quadratic* means "second-degree.") A quadratic equation looks like $ax^2 + bx + c = 0$. "Factorable" quadratic equations are solved by factoring and setting each factor equal to zero. The solutions are called *roots*

of the quadratic, values of the variable that satisfy the equation.

Example:

Solve $x^2 - 5x - 14 = 0$.

$$x^2 - 5x - 14 = (x - 7)(x + 2) = 0 \quad \text{Factor the trinomial.}$$

$$(x - 7) = 0 \quad \text{or} \quad (x + 2) = 0 \quad \text{Set each factor equal to zero.}$$

$$x = 7 \quad \text{or} \quad -2$$

The solution set is $\{7, -2\}$. Answer

Therefore, the *roots* of the quadratic equation $x^2 - 5x - 14 = 0$ are 7 and -2, since $7^2 - 5 \times 7 - 14 = 0$ and $(-2)^2 - 5 \times (-2) - 14 = 0$.

Quadratic Formula

As mentioned, the Quadratic Formula can be used to solve a quadratic equation, $ax^2 + bx + c = 0$, whether it's factorable or not. The formula uses the coefficients a , b , and c to find the solutions.

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a \neq 0$$

Remember the trinomial equation $x^2 - 8x + 9 = 0$ that we determined was “unfactorable.” Let's take a look at how to solve it using the Quadratic Formula.

Example:

Solve $x^2 - 8x + 9 = 0$.

Substitute $a = 1$, $b = -8$, and $c = 9$ into the Quadratic Formula.

$$x = \frac{[-b \pm \sqrt{(b^2 - 4ac)}]}{2a} = \frac{-(-8) \pm \sqrt{((-8)^2 - 4 \times 1 \times 9)}}{2 \times 1}$$

$$x = \frac{[8 \pm \sqrt{(64 - 36)}]}{2}$$

$$x = \frac{(8 \pm \sqrt{28})}{2} \quad \text{Simplify } \sqrt{28}.$$

$$x = \frac{(8 \pm 2\sqrt{7})}{2} \quad \text{Divide the numerator by 2.}$$

$$x = 4 \pm \sqrt{7} \quad \text{Answer}$$

The Quadratic Formula can also be used to solve “factorable” quadratic equations. Earlier, we determined that the equation $x^2 - 5x - 14 = 0$ had roots of 7 and -2. Let’s try to solve it a different way using the Quadratic Formula.

Example:

Solve $x^2 - 5x - 14 = 0$ using the Quadratic Formula.

$$a = 1, b = -5, c = -14$$

$$x = \frac{[-b \pm \sqrt{(b^2 - 4ac)}]}{2a} = \frac{-(-5) \pm \sqrt{((-5)^2 - 4 \times 1 \times -14)}}{2 \times 1}$$

$$x = \frac{[5 \pm \sqrt{(25 + 56)}]}{2}$$

$$x = \frac{(5 \pm \sqrt{81})}{2}$$

$$x = \frac{(5 \pm 9)}{2}$$

$$x = \frac{(5 + 9)}{2} = \frac{14}{2} = 7$$

$$x = \frac{(5 - 9)}{2} = \frac{-4}{2} = -2$$

The solution set is $\{7, -2\}$. Answer

Of course, the roots will be the same regardless of which method you choose to use to solve the quadratic equation.

Solving by Substitution

Sometimes equations that don't look like quadratic equations can be solved by factoring or by using the Quadratic Formula. This is true if you can rewrite the equations in quadratic form using substitution.

$$x^4 - 8x^2 - 11 = 0 \quad \rightarrow \text{Let } u = x^2, u^2 - 8u - 11 = 0$$

$$x + 2\sqrt{x} + 5 = 0 \quad \rightarrow \text{Let } u = \sqrt{x}, u^2 + 2u + 5 = 0$$

$$\left(\frac{1}{8x}\right)^2 + 3\left(\frac{1}{8x}\right) - 7 = 0 \quad \rightarrow \text{Let } u = \left(\frac{1}{8x}\right), u^2 + 3u - 7 = 0$$

As in the examples above, choose a value for u so that the equation becomes quadratic in u and fits the form $au^2 + bu + c = 0$.

Example:

Solve $x - 19\sqrt{x} + 48 = 0$.

Let $u = \sqrt{x}$ and the equation becomes $u^2 - 19u + 48 = 0$. Is this factorable? It turns out that it is. If you're unsure, use the Quadratic Formula to find the roots.

$$u^2 - 19u + 48 = (u - 16)(u - 3) = 0$$

$$(u - 16) = 0 \text{ or } (u - 3) = 0$$

$$u = 16 \text{ or } u = 3$$

$$\sqrt{x} = 16 \text{ or } \sqrt{x} = 3$$

$$x = 256 \text{ or } x = 9$$

You're not done yet! Substitute \sqrt{x} back in for u to find x .

{256, 9} Answer

The Discriminant

The *discriminant* of a quadratic equation equals $b^2 - 4ac$, the radicand in the Quadratic Formula. It allows you to determine the nature of the roots of a quadratic equation without actually solving for them.

1. If $b^2 - 4ac > 0$, there are two real, unequal roots.

When $b^2 - 4ac$ is a perfect square, there are *two real, rational roots*.

When $b^2 - 4ac$ is not a perfect square, there are *two real, irrational*

roots.

2. If $b^2 - 4ac = 0$, there is one real root. (It's called a *double root*.)
3. If $b^2 - 4ac < 0$, there are *no real roots*. (They're complex conjugate in the form $a + bi$ and $a - bi$.)

Example:

Determine the nature of the roots of the quadratic equation $x^2 - 9x - 10 = 0$.

$$\begin{aligned} b^2 - 4ac &= (-9)^2 - 4(1)(-10) \\ &= 81 + 40 = 121 \end{aligned} \quad \text{The discriminant is positive and a perfect square.}$$

Two real, rational roots Answer

Equations with Radicals

Radical equations contain radicals with variables in the radicand. To solve a radical equation, you must always isolate the radical first. Then raise both sides to the appropriate power to “undo” the radical—that is, square a square root, cube a cube root, and so on.

Example 1:

$$\begin{aligned} \sqrt{x} &= 11 \\ (\sqrt{x})^2 &= 11^2 \\ x &= 121 \end{aligned}$$

Example 2:

$$\begin{aligned} \sqrt{x+1} &= 3 \\ (\sqrt{x+1})^2 &= 3^2 \\ x+1 &= 9, \text{ so } x = 8 \end{aligned}$$

Example:

Solve $2\sqrt{x-12} - 10 = \sqrt{x-12}$.

$$\sqrt{x-12} = 10 \quad \text{Combine like terms.}$$

$$(\sqrt{x-12})^2 = 10^2 \quad \text{Isolate the radical and square both sides.}$$

$$x - 12 = 100$$

$$x = 112 \quad \text{Answer}$$

It is crucial to go back and check your answer in the original radical equation.

Sometimes squaring both sides of an equation introduces *extraneous roots* that do not satisfy the original equation. When $x = 112$, $2\sqrt{112-12} - 10$ equals $\sqrt{112-12}$, so the answer checks.

When there is more than one radical in an equation, you need to square both sides multiple times. Again, remember to isolate one radical at a time.

Example:

Solve $\sqrt{x+4} + \sqrt{x} - 6 = 0$.

$$\sqrt{x+4} = 6 - \sqrt{x} \quad \text{Isolate the radical } \sqrt{x+4}.$$

$$(\sqrt{x+4})^2 = (6 - \sqrt{x})^2 \quad \text{Square both sides.}$$

$$x + 4 = 36 - 12\sqrt{x} + x$$

$$-32 = -12\sqrt{x} \quad \text{Isolate the second radical.}$$

$$\frac{32}{12} = \frac{8}{3} = \sqrt{x}$$

$$\left(\frac{8}{3}\right)^2 = (\sqrt{x})^2 \quad \text{Square both sides a second time.}$$

$$\frac{64}{9} = x \quad \text{Answer}$$

Remember to check your answer in the original radical equation.

Inequalities

The rules for solving equations also apply to inequalities. Remember that multiplying or dividing both sides of an inequality by a negative number reverses its sign.

$$-x > 8 \text{ becomes } x < -8.$$

$$-\frac{1}{2}x < 15 \text{ becomes } x > -30.$$

Transitive Property of Inequality

The *Transitive Property of Inequality* states that for any real numbers a , b , and c :

If $a < b$ and $b < c$, then $a < c$.

It makes sense that if $3 < 4$ and $4 < 5$, then $3 < 5$.

Sometimes the transitive property is written by combining the three inequalities:

$$a < b < c \text{ and } 3 < 4 < 5$$

Addition and Multiplication Properties

The addition and multiplication properties of inequality mirror those for equalities.

<p>Addition Property of Inequality</p> <p>If $a < b$, then $a + c < b + c$.</p>	<p>If $40 < 50$, then $40 + 1 < 50 + 1$.</p>
--	--

<p>Multiplication Property of Inequality</p> <p>If $a < b$ and c is <u>positive</u>, then $ac < bc$.</p> <p>If $a < b$ and c is <u>negative</u>, then $ac > bc$.</p>	<p>If $5 < 6$, then $5(2) < 6(2)$.</p> <p>If $5 < 6$, then $5(-2) > 6(-2)$.</p>
---	---

Example:

Solve $2 - x > 4 + x$.

$$-2 > 2x$$

$$-1 > x$$

$$\{x: x < -1\} \quad \text{Answer}$$

This reads “the set of all x such that x is less than negative one.” On a number line, the graph would begin with an open circle at -1 and extend forever in the left direction.

“And” vs. “Or”

A *conjunction* joins two sentences with “and” and is true when *both* sentences are true. For instance:

$$x > 29 \quad \text{and} \quad x < 48$$

$29 < x < 48$ is an alternate way of writing this, and it’s read “ x is greater than 29 and less than 48.”

A *disjunction* joins two sentences with “or” and is true when *at least one* of the sentences is true. For instance:

$$x > 5 \quad \text{or} \quad x < -5$$

Example:

Solve the conjunction $0 < x - 2 \leq 6$.

Rewrite the inequality using “and.”

$$0 < x - 2 \quad \text{and} \quad x - 2 \leq 6$$

$$2 < x \quad \text{and} \quad x \leq 8$$

The solution set is $\{x: 2 < x \leq 8\}$. Answer

On a number line, the graph would begin with an open circle at 2, extend to the right, and end with a closed circle at 8.

Inequalities with Absolute Value

Inequalities involving absolute value can be thought of as either disjunctions or conjunctions.

- $|x| < n$ is equivalent to the *conjunction* $-n < x < n$.
 $|x| < 2$ is equivalent to the conjunction $-2 < x < 2$.
- $|x| > n$ is equivalent to the *disjunction* $x < -n$ or $x > n$.
 $|x| > 2$ is equivalent to the disjunction $x < -2$ or $x > 2$.

Example:

Solve $8 - |3x + 7| < 2$.

$$-|3x + 7| < -6$$

$$|3x + 7| > 6$$

Divide both sides by -1 , reversing the sign.

Analyze the inequality to determine if it's a conjunction or disjunction. Since the variable is on the greater than side of the inequality sign, it resembles the second example above $|x| > n$. The inequality is therefore a *disjunction*.

$$3x + 7 > 6 \quad \text{or} \quad 3x + 7 < -6$$

$$3x > -1 \quad \text{or} \quad 3x < -13$$

$$x > -\frac{1}{3} \quad \text{or} \quad x < -\frac{13}{3}$$

$$\left\{ x : x < -\frac{13}{3} \text{ or } x > -\frac{1}{3} \right\} \quad \text{Answer}$$

Rational Expressions

Simplifying Rational Expressions

Rational numbers are numbers that can be expressed as $\frac{p}{q}$, a quotient of integers. Likewise, *rational expressions* are those that can be expressed as a quotient of polynomials. $\frac{(x^2 - 9)}{x}$, $\frac{(x^2 - 10x + 25)}{(x^2 + 100)}$, and $\frac{14xy}{xy^3}$ are

examples of rational expressions. The principles of factoring are important when you are simplifying rational expressions. Take a moment to review some special products of polynomials:

SPECIAL PRODUCTS

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Example:

Simplify $\frac{(2x^2 - 6x)}{x^2 - 9}$.

$$\frac{(2x^2 - 6x)}{x^2 - 9} = \frac{2x(x - 3)}{(x + 3)(x - 3)}$$

Factor the numerator and denominator. Recognize that the numerator has a common factor of $2x$ and the denominator is the difference of perfect squares. Now simplify by dividing the numerator and denominator by their common factor $(x - 3)$.

$$\frac{2x}{x + 3}, x \neq 3 \quad \text{Answer}$$

Typically, rational expressions are in simplest form when they do not contain zero or negative exponents. Recall that any number raised to a zero power equals one and any number raised to a negative one power equals its reciprocal.

Multiplying and Dividing Rational Expressions

Multiplying and dividing rational expressions follow the same rules as fractions.

Multiplication Rule for Fractions

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad b, d \neq 0$$

Division Rule for Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}, \quad b, c, d \neq 0$$

Example:

$$\frac{4}{x^2} \div \frac{x}{4}$$

$$\frac{4}{x^2} \div \frac{x}{4} = \frac{4}{x^2} \times \frac{4}{x} \quad \text{Multiply by the reciprocal of } \frac{x}{4}.$$

$$= \frac{16}{x^3} \quad \text{Answer}$$

Example:

$$\text{Simplify } \frac{(x^4 - y^4)}{(x^2 + y^2)}(x + y)^{-2}.$$

Factor the first fraction. The numerator $x^4 - y^4$ is the difference of perfect squares: $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$. The denominator $x^2 + y^2$ is the sum of perfect squares and therefore cannot be factored. The simplified expression is

$$\frac{(x^2 + y^2)(x^2 - y^2)}{(x^2 + y^2)}(x + y)^{-2}$$

Simplify the fraction by factoring out a common factor of $x^2 + y^2$:

$$\frac{(x^2 - y^2)}{1}(x + y)^{-2}$$

Simplify the negative exponent by rewriting $(x + y)^{-2}$ as $\frac{1}{(x + y)^2}$. Now you

have:

$$\frac{(x^2 - y^2)}{1} \times \frac{1}{(x + y)^2}$$

The numerator of the first term is the difference of perfect squares and can be factored. In factored form, you have

$$\frac{(x+y)(x-y)}{1} \times \frac{1}{(x+y)(x+y)}$$

Factor out a common factor of $(x+y)$.

$$= \frac{(x-y)}{(x+y)} \quad \text{Answer}$$

This example is about as complex as you'll find on the Level 1 test. It incorporates many concepts reviewed thus far: factoring the difference of perfect squares, simplifying rational expressions, finding the greatest common factor, multiplying rational expressions, and simplifying negative exponents.

Adding and Subtracting Rational Expressions

Similar to fractions, rational expressions can only be added or subtracted when they have a common denominator. The *least common denominator (LCD)* is the *least common multiple (LCM)* of the denominators.

Example:

$$\text{Simplify } \frac{1}{8x} - \frac{-1}{4x^2}.$$

The LCM of $8x$ and $4x^2$ is $8x^2$. Express each fraction as an equivalent fraction with a denominator of $8x^2$.

$$\frac{1}{8x} \times \frac{x}{x} - \frac{1}{-4x^2} \times \frac{2}{2}$$

Now simplify. Change the subtraction to adding the opposite of the second term.

$$\frac{x}{8x^2} + \frac{2}{8x^2}$$

$$= \frac{(x+2)}{8x^2} \quad \text{Answer}$$

Example:

Simplify $\frac{2}{(x-1)^2} - \frac{2}{(1-x^2)}$.

Recall that $1-x^2 = -(x^2-1)$. The equation then becomes

$$\frac{2}{(x-1)^2} + \frac{2}{(x^2-1)}$$

The denominators can be factored as $(x-1)(x-1)$ and $(x+1)(x-1)$, resulting in an LCM of $(x+1)(x-1)^2$. Expressing each fraction with the LCM as its denominator gives you

$$\begin{aligned} & \frac{2(x+1)}{(x+1)(x-1)^2} + \frac{2(x-1)}{(x^2-1)(x-1)} \\ &= \frac{[2(x+1) + 2(x-1)]}{(x+1)(x-1)^2} \\ &= \frac{(2x+2+2x-2)}{(x+1)(x-1)^2} \\ &= \frac{4x}{(x+1)(x-1)^2} \quad \text{Answer} \end{aligned}$$

Solving Equations with Rational Expressions

One way to solve equations involving rational expressions is to multiply both sides by the least common denominator (LCD) of the rational expressions.

Example:

Solve $\frac{1}{(x+2)(x-7)} + 1 = \frac{1}{(x-7)}$.

The LCD is $(x+2)(x-7)$. Multiplying both sides by the LCD gives you:

$$1 + (x + 2)(x - 7) = (x + 2)$$

$$1 + x^2 - 5x - 14 = x + 2$$

$$x^2 - 6x - 15 = 0$$

Since this equation is not factorable, use the Quadratic Formula to find its roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(6 \pm \sqrt{96})}{2} = \frac{(6 \pm 4\sqrt{6})}{2}$$

$$x = 3 \pm 2\sqrt{6} \quad \text{Answer}$$

When multiplying both sides of an equation by an LCD, extraneous roots may be introduced. Always go back and check your answer in the original equation.

Systems

A *system of linear equations* is made up of two or more linear equations in the same two variables. Examples of systems are as follows:

Example 1:

$$\begin{cases} 2x - 4y = 10 \\ 6x + y = -3 \end{cases}$$

Example 2:

$$\begin{cases} y = 8x - 14 \\ y = x \end{cases}$$

Example 3:

$$\begin{cases} a + b = 37 \\ 5a - 2b = -13 \end{cases}$$

Systems of linear equations are also called *simultaneous equations*. The *solution* of a system is any ordered pair of the variables that satisfy all equations. For instance, in Example 2 above, the ordered pair (2, 2) is a solution of the system. When $x = 2$ and $y = 2$, both equations ($2 = 8(2) - 14$ and $2 = 2$) are satisfied. *Equivalent systems* are systems that have the same solution.

Three methods to solve systems are (1) substitution, (2) linear combination, and (3) graphing. Here we'll explain the first two methods. Graphing is discussed in the Coordinate Geometry chapter.

Solving by Substitution

Just as one equation allows you to solve for one unknown variable, two equations allow you to solve for two unknowns. The substitution method involves expressing one variable in terms of the other in one equation (i.e., x in terms of y or y in terms of x) and *substituting* this value into the second equation.

Example:

Solve the system using the substitution method:

$$\begin{cases} x + 3y = 14 \\ 3x - 2y = -2 \end{cases}$$

Choose one equation and solve for a variable. (You can choose what equation to use and for what variable to solve.) Let's solve for x in the first equation. This is the best choice, since the coefficient of x is 1.

$$\begin{cases} x = -3y + 14 \\ 3x - 2y = -2 \end{cases}$$

Substitute $-3y + 14$ for x in the second equation and solve for y :

$$\begin{aligned} 3(-3y + 14) - 2y &= -2 \\ -9y + 42 - 2y &= -2 \\ -11y &= -44 \\ y &= 4 \end{aligned}$$

Substitute 4 for y in either of the original equations in the system to solve for x :

$$x + 3(4) = 14$$

$$x + 12 = 14$$

$$x = 2$$

The solution is the ordered pair (2, 4). Answer

It is a good idea to check your answer by substituting (2, 4) back into both of the equations.

Solving by Linear Combination

Adding two linear equations results in what's called a *linear combination* of the equations. For example:

$$\begin{array}{r} x + 2y = 7 \\ +3x + y = -8 \\ \hline 4x + 3y = -1 \end{array}$$

The linear combination method involves transforming and adding equations in order to eliminate one variable and solve for the other. The goal is to end up with one equation in one variable. Let's solve the previous example in a different way.

Example:

Solve the system using the linear combination method:

$$x + 3y = 14$$

$$3x - 2y = -2$$

Let's try to eliminate x and solve for y . (You can choose to eliminate either variable here.) Start by multiplying the first equation by -3 so the coefficients of the x are opposites. Remember, what you do to one side of an equation you *must* do to the other, so both sides need to be multiplied by a -3 factor.

$-3(x + 3y = 14)$ becomes $-3x + -9y = -42$

This results in the equivalent system:

$$-3x + -9y = -42$$

$$3x - 2y = -2$$

Add the two equations:

$$\begin{array}{r} -3x + -9y = -42 \\ + \quad 3x - 2y = -2 \\ \hline 0x + 11y = -44 \end{array}$$

It is important that one variable cancels out. If this doesn't happen, check your work for errors or try multiplying by a different number in the first step. Now that x is eliminated, you are able to solve for y .

$$-11y = -44$$

$$y = 4$$

Substitute 4 for y in either of the original equations in the system to solve for x :

$$3x - 2(4) = -2$$

$$3x - 8 = -2$$

$$3x = 6$$

$$x = 2$$

The solution is the ordered pair $(2, 4)$. Answer

This, of course, is the same answer you got by using the substitution method. Remember that you have a choice of what method to use when solving systems of linear equations.

No Solution vs. Infinite Solutions

Systems of linear equations can have three possible solution sets:

1. One solution
2. No solution
3. Infinitely many solutions

Graphing systems will be discussed in the Coordinate Geometry chapter, but it is worth mentioning that one solution occurs when the two lines intersect in one point; no solution occurs when the lines are parallel; and infinitely many solutions occur when the lines are actually the same line.

When you are solving a system algebraically, *no solution* results from a *contradiction*. You end up with a statement that will never be true, such as $5 = 6$ or $-1 = 0$.

Example:

Solve the system:

$$2y = x + 36$$

$$y = \frac{1}{2}x + 4$$

Since the second equation is already solved for y in terms of x , let's substitute that value into the first equation:

$$2\left(\frac{1}{2}x + 4\right) = x + 36$$

$$x + 8 = x + 36$$

$$8 = 36$$

Of course, $8 \neq 36$, so this contradiction shows the systems has no solution.

No solution Answer

This example can be solved an alternate way by rewriting the first equation in

slope-intercept form. This results in the system:

$$y = \frac{1}{2}x + 18$$

$$y = \frac{1}{2}x + 4$$

Immediately you can see that these lines have the same slope, $m = \frac{1}{2}$.

Therefore, they must be parallel and will, by definition, never intersect.

Infinitely many solutions, on the other hand, result from an **identity**. You end up with a statement that is always true, such as $7 = 7$ or $0 = 0$.

Example:

Solve the system:

$$\begin{aligned}y &= 3x - 9 \\ -6x + 2y &= -18\end{aligned}$$

Let's use the linear combination method to solve this system. Start by rewriting the first equation:

$$\begin{cases} -3x + y = -9 \\ -6x + 2y = -18 \end{cases}$$

Multiply the first equation by -2 so the coefficients of the y are opposites:

$$\begin{cases} 6x + -2y = 18 \\ -6x + 2y = -18 \end{cases}$$

Add the equations:

$$\begin{array}{r}
 6x + -2y = 18 \\
 + \quad -6x + \quad 2y = -18 \\
 \hline
 0x + \quad 0y = 0 \\
 \qquad \qquad 0 = 0
 \end{array}$$

Zero always equals zero, so this identity shows the system has infinitely many solutions.

Infinitely many solutions Answer

An alternate method of solving the system is to rewrite the second equation, isolating y on the left side:

$$\begin{aligned}
 y &= 3x - 9 \\
 2y &= 6x - 18
 \end{aligned}$$

Both of these lines have a slope of 3 and a y -intercept of 9, so they are, in fact, the same line. You may also notice that the second equation is equivalent to the first. It is simply a factor of 2 greater than $y = 3x - 9$.

Word Problems with Systems

Word problems with two unknowns can be solved by setting up a system and then solving it using either the substitution or linear combination method.

Example:

Tickets for the homecoming football game cost \$3 for students and \$5 for the general public. Ticket sales totaled \$1,396, and 316 people attended the game. How many student tickets were sold?

First, define the variables. Let s = the number of student tickets and p = the number of tickets for the general public.

Since ticket sales totaled \$1,396:

$$3s + 5p = 1,396$$

Now use the given information on attendance to get:

$$s + p = 316$$

The system looks like:

$$\begin{cases} 3s + 5p = 1,396 \\ s + p = 316 \end{cases}$$

Which method, substitution or linear combination, works best here? Both produce the same solution, so it is your choice. Let's try using the linear combination method. Multiply the second equation by -3 and add to get

$$\begin{array}{r} 3s + 5p = 1,396 \\ + \quad -3s + -3p = -948 \\ \hline 0s + 2p = 448 \\ \quad p = 224 \end{array}$$

Substitute $p = 224$ into one of the original equations:

$$s + 224 = 316$$

$$s = 92$$

92 student tickets were sold. Answer

As with all word problems, always make sure to answer the question asked. Although we solved for both p and s , the word problem asks for the number of student tickets.

Example:

A mother is twice as old as her daughter. Twelve years ago she was three times as old as her daughter was then. Find the mother's present age.

Let m = the mother's age now and d = the daughter's age now. Since the mother is twice as old as her daughter, you have the first equation:

$$m = 2d$$

Twelve years ago, the mother was $m - 12$ years old and the daughter was $d - 12$ years old. This gives you the second equation:

$$m - 12 = 3(d - 12)$$

$$m - 12 = 3d - 36$$

$$m = 3d - 24$$

You have two equations in two unknowns, m and d . A system is needed to solve the word problem:

$$m = 2d$$

$$m = 3d - 24$$

Again, you can choose to solve by either the substitution or linear combination method. Let's try using substitution. Since the first equation is already solved for m in terms of d , substitute $m = 2d$ into the second equation and solve for d .

$$2d = 3d - 24$$

$$24 = d$$

The word problem asks for the mother's age, so substitute $d = 24$ back into one of the original equations.

$$m = 2(24)$$

$$m = 48$$

The mother is 48 years old. Answer

REVIEW QUESTIONS

1. If $x = 3y$ and $y = -2$, what is the value of $\frac{1}{2}x$?

- (A) -6
- (B) -3
- (C) -1
- (D) 3
- (E) 6

2. Solve $10 \times 3 - 1(2^3 \div 4) \times -5 \div \frac{2}{5} + 10$

- (A) -615
- (B) -340
- (C) 44
- (D) 65
- (E) 109

3. Evaluate the expression $|-x + 3| + y = 6$, if $y = -2$

- (A) $x = -5$ or $x = -11$
- (B) $x = 5$ or $x = -11$
- (C) $x = -5$ or $x = 11$
- (D) $x = -8$ or $x = 5$
- (E) $x = 5$ or $x = 11$

4. What are the roots of $2x^2 - 17x - 21 = 0$?

- (A) $\frac{3}{2}$ and 7
- (B) $\frac{-3}{2}$ and 7
- (C) $\frac{3}{2}$ and -7
- (D) $\frac{-3}{2}$ and -7
- (E) 7 and -7

5. Isabel leaves a $2\frac{1}{2}$ hour class after 45 minutes. What fraction of the class does she miss?

- (A) $\frac{1}{18}$
- (B) $\frac{3}{10}$
- (C) $\frac{1}{3}$
- (D) $\frac{7}{10}$
- (E) $\frac{2}{3}$

6. If a is a positive integer such that $4^a < 1,024^4$, what is the largest possible value of a ?

- (A) 10
- (B) 12

(C) 14

(D) 15

(E) 16

7. $\frac{8 + 4\sqrt{5}}{14 + 6\sqrt{5}} =$

(A) $\frac{1}{4}$

(B) $\frac{\sqrt{5} - 1}{2}$

(C) $\frac{\sqrt{5} + 1}{2}$

(D) $\frac{\sqrt{5}}{2}$

(E) $\frac{4}{5}$

8. Solve $2x^2 - 3x - 4 = 0$

(A) $\frac{3 \pm \sqrt{41}}{4}$

(B) $\frac{-3 \pm \sqrt{35}}{4}$

(C) $\frac{3 \pm \sqrt{41}}{2}$

(D) $\frac{-3 \pm \sqrt{41}}{4}$

(E) $\frac{3 \pm \sqrt{35}}{4}$

9. Movie ticket prices are \$5.00 for children and \$7.25 for adults. On Tuesday, the movie theater sold 390 tickets and made \$2,719.50. How many children's tickets were sold?
- (A) 240
(B) 204
(C) 192
(D) 156
(E) 48
10. Phone service costs a flat rate of \$30.00 per month, plus a one-time setup fee of \$20.00. A tax of 8.25% applies to the monthly rate. What is the total cost of new phone service for 3 months?
- (A) \$54.13
(B) \$94.43
(C) \$117.43
(D) \$119.08
(E) \$162.39

ANSWERS AND EXPLANATIONS

1. (B) Substitute -2 for y .

$$x = 3y$$

$$x = 3(-2)$$

$$x = -6$$

The question asks for $\frac{1}{2}x$, so $\frac{1}{2}(-6) = -3$

2. (D) Be careful to follow PEMDAS, the order of operations.

$$10 \times 3 - 1(8 \div 4) \times -5 \div \frac{2}{5} + 10$$

$$10 \times 3 - 1(2) \times -5 \div \frac{2}{5} + 10$$

$$30 - 1(2) \times -5 \div \frac{2}{5} + 10$$

$$30 - 2 \times -5 \div \frac{2}{5} + 10$$

$$30 - (-10) \div \frac{2}{5} + 10$$

$$30 - (-25) + 10$$

$$55 + 10$$

$$65$$

3. (C) Substitute -2 for y .

$$|-x + 3| + y = 6$$

$$|-x + 3| + (-2) = 6$$

$$|-x + 3| = 8$$

$$-x + 3 = 8 \text{ or } -x + 3 = -8$$

$$x = -5 \text{ or } x = 11$$

4. (A) Use the FOIL method.

$$2x^2 - 17x + 21 = 0$$

$$(2x - 3)(x - 7) = 0$$

$$\text{So } 2x - 3 = 0$$

$$2x = 3 \text{ or } x - 7 = 0$$

$$x = \frac{3}{2} \text{ or } x = 7$$

5. (D) Convert hours to minutes. $2\frac{1}{2}$ hours = 150 min. She missed 150 - 45 = 105 min.

$$\frac{105}{150} = \frac{7}{10}$$

6. (D) Rewrite using base 4. $4^a < 1,024^4 = 4^a < (4^4)^4 = 4^a < 4^{16}$

Since a is an integer, the largest a can be is 15.

7. (B) Use the identity property to multiply by 1 in the form $\frac{14 - 6\sqrt{5}}{14 - 6\sqrt{5}}$

$$\frac{8 + 4\sqrt{5}}{14 + 6\sqrt{5}} \times \frac{14 - 6\sqrt{5}}{14 - 6\sqrt{5}} = \frac{112 - 48\sqrt{5} + 56\sqrt{5} - 120}{196 - 180}$$

$$\text{Simplify: } \frac{-8 + 8\sqrt{5}}{16} = \frac{8(-1 + \sqrt{5})}{16} = \frac{-1 + \sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2}$$

8. (A) Use the quadratic formula: $a = 2$, $b = -3$, $c = -4$

$$\frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9 + 32}}{4}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

9. (E) Set up a system of equations.

Let x = the number of child movie tickets and y = the number of adult movie tickets.

$$5x + 7.25y = 2,719.50 \text{ and } x + y = 390$$

Multiply the second equation by -5 .

$$-5(x + y = 390) = -5x - 5y = -1,950$$

Add this to the first equation.

$$\begin{array}{r} -5x - 5y = -1,950 \\ +5x + 7.25y = 2,719.50 \\ \hline 2.25y = 769.50 \end{array}$$

$y = 342$ adult tickets. $390 - 342 = 48$ child tickets.

10. (C) Set up an equation to find the monthly rate.

\$30 per month \times 8.25% tax: $30(1.0825)$

Now multiply that by 3 months: $3[30(1.0825)]$

Now add the \$20 setup fee: $3[30(1.0825)] + 20$

Calculate, using order of operations (PEMDAS):

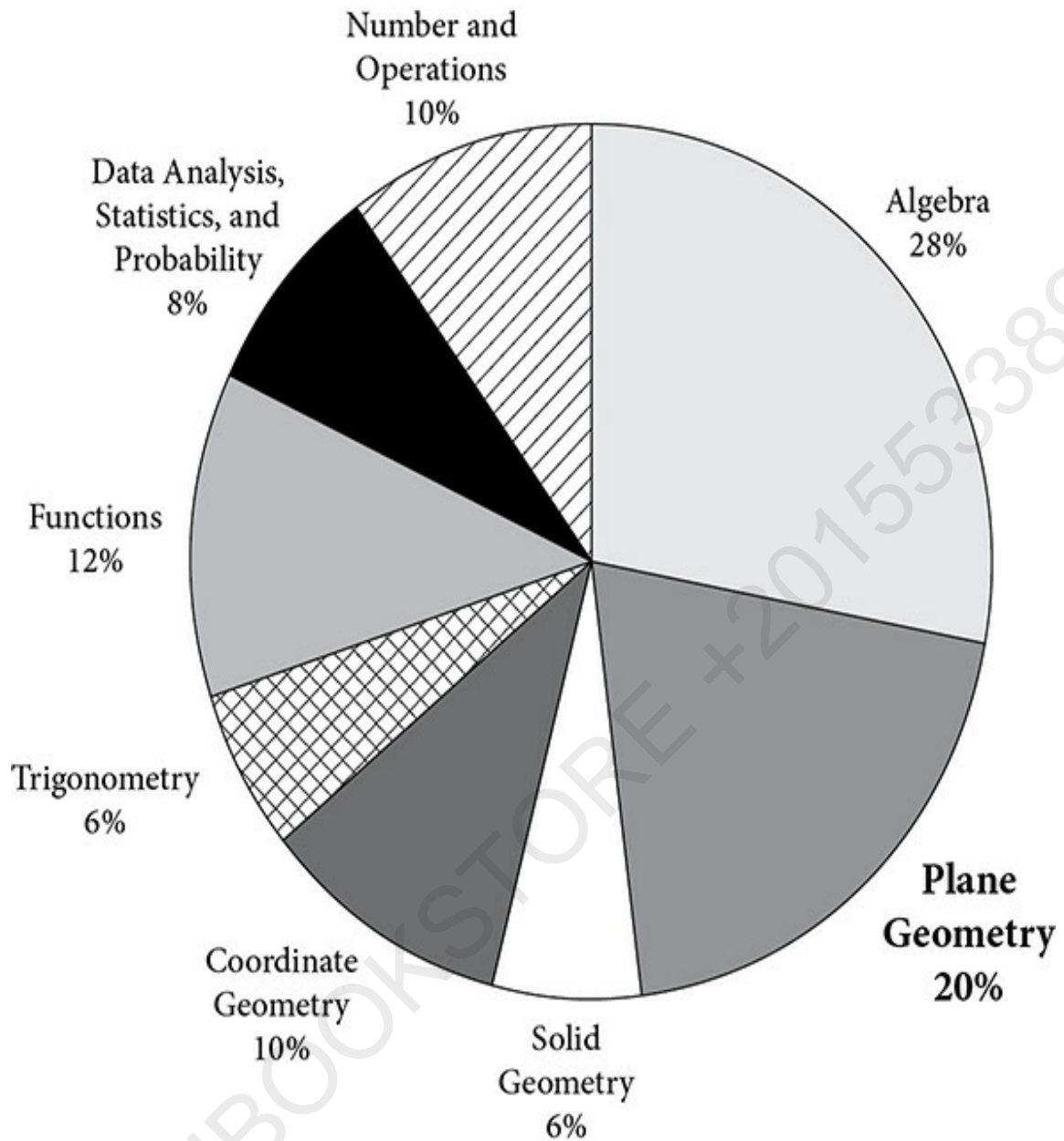
$$3(32.475) + 20 = 97.425 + 20 = 117.425 = \$117.43$$

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Plane Geometry

This chapter provides a review of Plane Euclidean Geometry principles. Eighteen to 22 percent of the Level 1 test questions relate specifically to plane geometry. By definition, *plane geometry* focuses on two-dimensional shapes and figures. Solid (i.e., three-dimensional) and coordinate geometry are covered in the succeeding chapters. The pie chart shows approximately how much of the Level 1 test is related to plane geometry:



The following geometry topics are covered in this chapter:

1. Undefined Terms
 - a. Points, Lines, Planes
2. Lines, Segments, Rays
3. Angles
 - a. Measures of Angles
 - b. Supplementary and Complementary Angles
 - c. Vertical Angles
 - d. Linear Pairs of Angles

4. Triangles
 - a. Types of Triangles
 - b. Sum of Interior Angles and Exterior Angles
 - c. Medians, Altitudes, and Angle Bisectors
 - d. Congruent Triangles
 - i. SSS, SAS, and ASA Postulates and the AAS Theorem
 - ii. CPCTC (Corresponding Parts of Congruent Triangles are Congruent)
 - e. Isosceles Triangles
 - i. Isosceles Triangle Theorem
 - f. Triangle Inequality
 - g. Pythagorean Theorem
 - h. Special Right Triangles
5. Parallel Lines
6. Polygons
 - a. Types of Polygons
 - b. Perimeter
 - c. Sum of the Interior Angles
 - d. Sum of the Exterior Angles
 - e. Special Quadrilaterals
7. Similarity
 - a. Ratio and Proportion
 - b. Similar Triangles
8. Circles
 - a. Chords
 - b. Tangents
 - c. Arcs and Angles
 - d. Circumference
 - e. Arc Length
9. Area

- a. Area Formulas
- b. Area vs. Perimeter
- c. Area Ratio of Similar Figures
- d. Figures That Combine Numerous Shapes

Undefined Terms

Three “undefined terms” make up the foundation of geometry:

1. **Point.** A location in space. A point has no size or dimension.

•P

2. **Line.** A collection of infinitely many points extending forever in opposite directions.



3. **Plane.** A flat surface that extends forever in all directions. A plane has no thickness.



Let's review some of the basic definitions for points, lines, and planes. *Collinear points* lie on the same line. Two distinct points are always collinear, and three points may or may not be collinear. *Coplanar points* lie on the same plane, and the set of all points is called *space*.

Following are a few key things to remember about the intersection of points, lines, and planes:

- Two distinct points determine a line.
- If two distinct lines intersect, then they intersect in a point.
- If two distinct planes intersect, then they intersect in a line.

- If a plane intersects a line, then they intersect in a point. (This assumes the line is not contained in the plane.)
- Three noncollinear points determine a plane. This can be expanded upon to show that two intersecting lines determine a plane. A line and a point not on it also determine a plane.

Example:

Each of the following sets of points could be the intersection of two planes EXCEPT:

- (A) the empty set
- (B) a point
- (C) a line
- (D) a plane
- (E) cannot be determined

The intersection of two planes is the set of points that they share. If two planes do not intersect, then their intersection is the empty set. If two planes are in fact the same plane, then their intersection is a plane. If two (distinct) planes intersect, then they intersect in a line. It is not possible for two planes to intersect in a point, so the correct answer is B.

B Answer

When in doubt, draw a picture to help solve geometry problems. Pictures allow you to visualize what's happening in a problem. The next example uses a picture to better understand how lines can be determined from four noncollinear points.

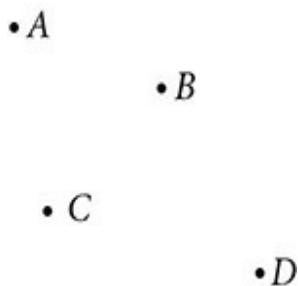
Example:

How many lines are determined by four noncollinear points?

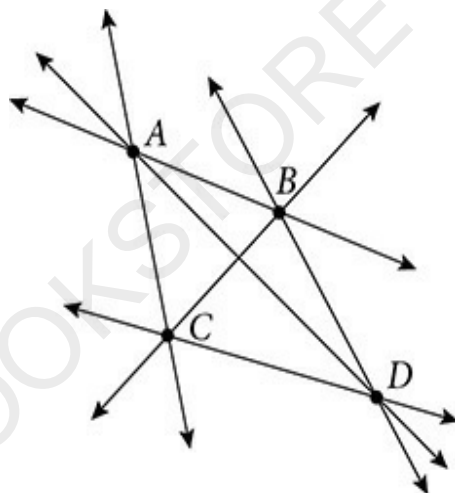
- (A) 2
- (B) 4

- (C) 6
- (D) 8
- (E) infinitely many

Two distinct points determine a line. Draw four noncollinear points to visualize the problem.



Draw lines through each pair of points as shown.



Six distinct lines are determined. Answer

Lines, Segments, Rays

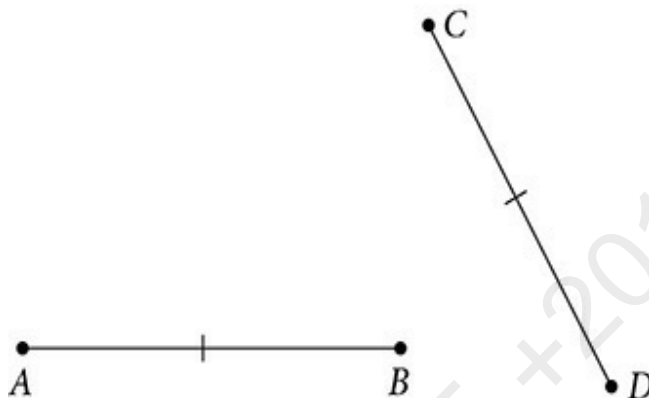
A *ray* has one endpoint and extends forever in one direction.



A *line segment* is the set of all points between two endpoints on a line.



Segments that have the same measure are called *congruent segments*. For example, if $\overline{AB} = \overline{CD}$, then \overline{AB} is congruent to \overline{CD} . In a mathematical expression, congruency is written as the symbol \cong , $\overline{AB} \cong \overline{CD}$. Congruency is depicted as “tick marks” in diagrams as shown below:



Given three collinear points X, Y, and Z:

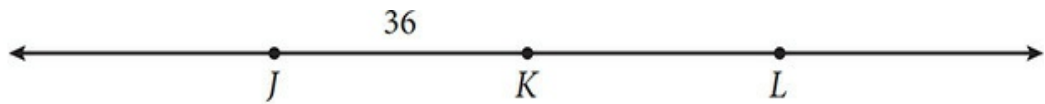


$\overline{XY} + \overline{YZ} = \overline{XZ}$. In other words, the distance from X to Y added to the distance from Y to Z equals the total distance from X to Z. This is called the *Segment Addition Postulate*. Y is between X and Z by the *definition of betweenness*. If $\overline{XY} = 2$ cm and $\overline{YZ} = 3$ cm, for example, it makes sense that \overline{XZ} should equal 5 cm.

Example:

Points J, K, and L are collinear. K is between J and L. If the length of $\overline{JK} = 36$ and the length of $\overline{JK} = \frac{3}{4} \overline{JL}$ what is the length of \overline{KL} ?

Drawing a diagram such as the one below will help to visualize the given information.



$$36 = \frac{3}{4} \overline{JL}$$

$$36 \times \frac{4}{3} = \overline{JL}$$

$\overline{JL} = 48$ cm. You're not done yet. Now use the definition of betweenness to find the length of \overline{KL} . Since the problem states that K is between J and L , you can conclude that $\overline{JK} + \overline{KL} = \overline{JL}$.

$$36 + \overline{KL} = 48$$

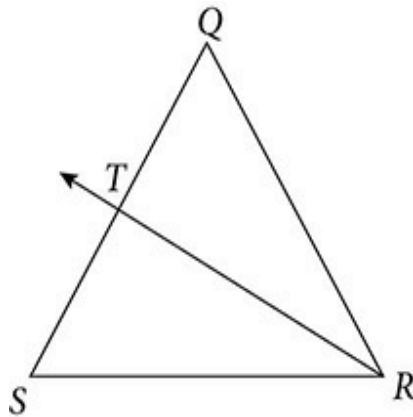
$$\overline{KL} = 12 \text{ cm} \quad \text{Answer}$$

The *Midpoint Theorem* states that if M is the midpoint of \overline{XY} , then $\overline{XM} = \frac{1}{2} \overline{XY}$. You can also say that

$$2\overline{XM} = \overline{XY}, 2\overline{MY} = \overline{XY}, \overline{MY} = \frac{1}{2} \overline{XY}, \text{ or } \overline{XM} = \overline{MY}.$$

Example:

$\triangle QRS$ is an equilateral triangle with a perimeter of 36. If line \overleftrightarrow{RT} bisects \overline{QS} , find \overline{QT} .



Since $\triangle QRS$ is equilateral, each side measures 12. Since \overleftrightarrow{RT} is the bisector of the side \overline{QS} point T must be the midpoint of the side. The

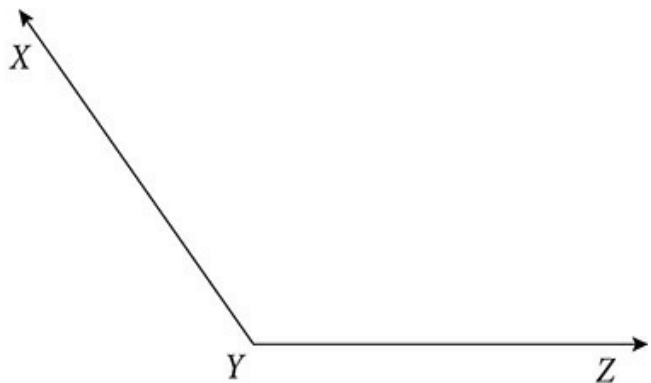
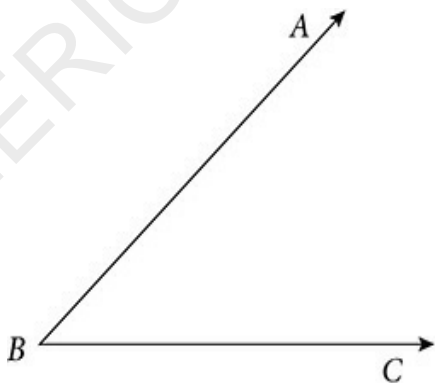
Midpoint Theorem tells you that $\overline{QT} = \frac{1}{2}\overline{QS}$.

$$\overline{QT} = \frac{1}{2} \times 12$$

$$\overline{QT} = 6 \quad \text{Answer}$$

Angles

An *angle* is the union of two noncollinear rays. The rays themselves are called the *sides*, and the shared endpoint is called the *vertex*. Some textbooks teach that the union of two collinear rays is called a straight angle. Since this is not a universal term,



Angles are typically named using three capital letters, such as $\angle ABC$ and $\angle XYZ$. The middle letter always represents the vertex of the angle. In cases where there's one unique angle from a given vertex, the angle can be named using one letter, such as $\angle B$ and $\angle Y$.

Measures of Angles

The *measure of an angle* is a unique number between 0° and 180° . Note that an angle's measure cannot equal 180° or 0° , since by definition an angle is the union of *noncollinear rays*. Think of a protractor when assigning a degree measure to an angle. A protractor only allows you to assign numbers between 0 and 180. The "measure of angle ABC equals 45 degrees" is written as $m\angle ABC = 45^\circ$.

Angles are classified by their measure:

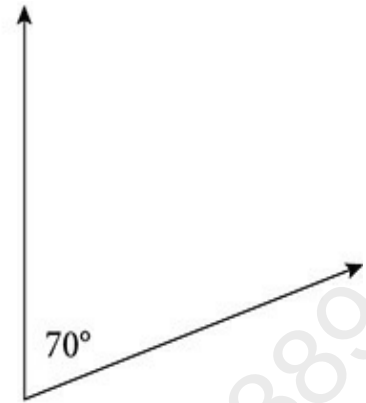
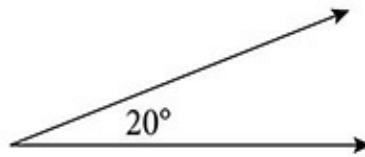
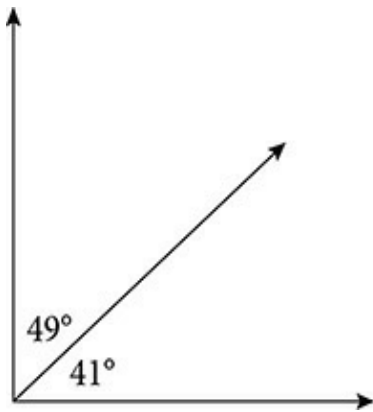
- **Right angles** measure 90° .
- **Acute angles** measure less than 90° .
- **Obtuse angles** measure greater than 90° but less than 180° .

If two lines intersect to form right angles, the lines are said to be *perpendicular*. The symbol for perpendicular is " \perp ". The expression $l_1 \perp l_2$, reads "line 1 is perpendicular to line 2." In diagrams, perpendicular lines are shown by a small box in the angle.

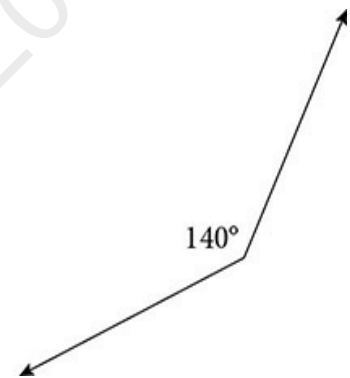
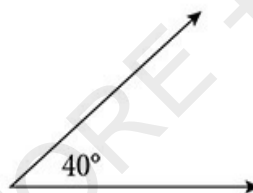
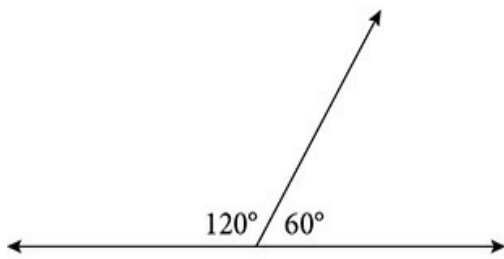
Remember that figures are not always drawn to scale. Just because an angle appears to be acute or lines appear to be perpendicular, don't assume this is true. Look at the given information in the problem to clarify figures. To answer some problems, it may be helpful to draw your own diagram.

Supplementary and Complementary Angles

Complementary angles are two angles whose measures add up to 90° . Complementary angles may or may not share a side. (If they do share a side, they are known as *adjacent angles*.)



Supplementary angles are two angles whose measures add up to 180° . Similar to complementary angles, supplementary angles may or may not share a side.



Example:

Three times the measure of the complement of a certain angle is equal to 40° less than the measure of the supplement of that angle. What is the measure of the angle?

Let x = the measure of the angle and $90 - x$ = the measure of its complement. It is important to define the given information in terms of *one* variable if possible. If you let x = the measure of the angle and y = the measure of its complement, you're left with $x + y = 90^\circ$, which is not enough information to solve for both variables. Remember that the sum of an angle and its complement is 90° so use that information when defining the second angle as $90 - x$. Similarly, let $180 - x$ = the measure of the angle's supplement. Set up the equation:

$$3(90 - x) = (180 - x) - 40$$

$$270 - 3x = 140 - x$$

$$130 = 2x$$

$$x = 65^\circ \quad \text{Answer}$$

The complement of the angle is 25° , since $65 + 25 = 90$.

The supplement of the angle is 115° , since $65 + 115 = 180$.

Example:

Each of the following statements is true EXCEPT:

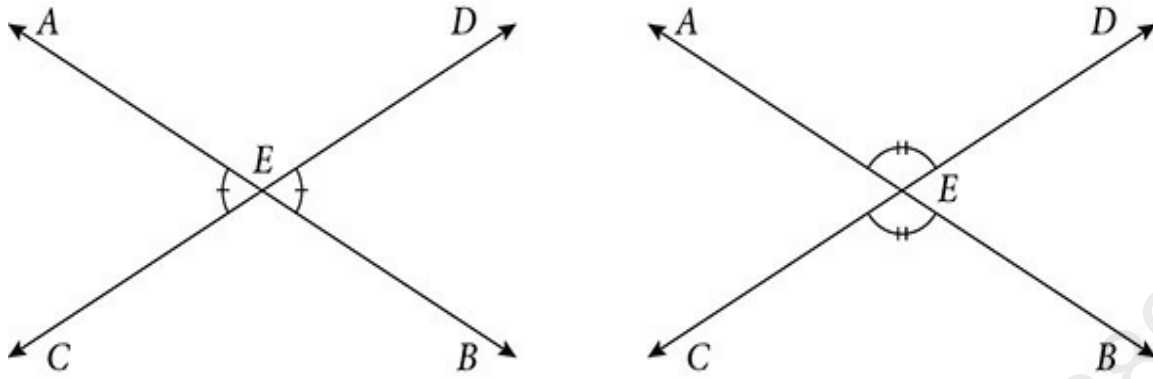
- (A) A complement of an acute angle is acute.
- (B) A supplement of an obtuse angle is acute.
- (C) The supplement of an acute angle is obtuse.
- (D) The complement of a right angle is a right angle.
- (E) The supplement of a right angle is a right angle.

The complement of an acute angle must always measure less than 90° , so A is true. Let's take an obtuse angle measuring 100° . Its supplement is 80° , which is an acute angle. B is true. Answer C states the opposite of B and is also true. D and E involve right angles. Since $90 + 90 = 180$, a right angle and another right angle are in fact supplementary, so answer E must be true. That leaves D. A right angle does not have a complement. $90 + 0 = 90$, and we've already determined that an angle cannot measure 0° .

The correct answer is D. Answer

Vertical Angles

Vertical angles are opposite angles formed by two intersecting lines. Vertical angles are always congruent, as shown by the congruency marks in the diagram.



Two pairs of vertical angles are formed when two lines intersect.
 $\angle AEC \cong \angle DEB$ and $\angle AED \cong \angle CEB$.

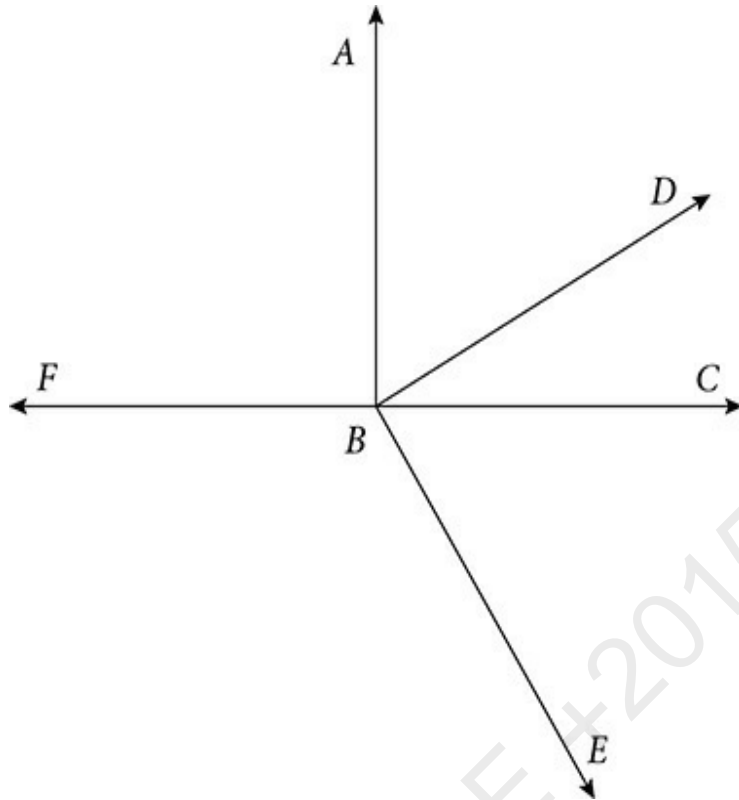
Linear Pairs of Angles

A *linear pair of angles* is formed by two angles that share a common side and whose noncommon sides form a straight line. By definition, linear pairs of angles are always adjacent angles and are also always supplementary.



Example:

In the figure below, $\angle ABC$ and $\angle DBE$ are right angles and the measure of $\angle EBF$ is four times the measure of $\angle DBC$. Find the measure of $\angle ABD$.



Let $m\angle DBC = x$ and $m\angle EBF = 4x$. It follows that the $m\angle ABD = 90 - x$ and $m\angle CBE = 90 - x$, since $\angle ABC$ and $\angle DBE$ are right angles. Note that $\angle CBE$ and $\angle EBF$ are a linear pair, and therefore, they are supplementary.

$$m\angle CBE + m\angle EBF = 180$$

$$(90 - x) + 4x = 180$$

$$90 + 3x = 180$$

$$3x = 90$$

$$x = 30$$

Since the question asks for the measure of $\angle ABD$:

$$m\angle ABD = 90 - x = 90 - 30 = 60^\circ$$

60° Answer

Note that the complements of the same angle are congruent. The measure of $\angle DBC$ is 30° while both its complements, $\angle ABD$ and $\angle CBE$, measure 60° . There's one unique number that will result in 90° when

added to 30° . In fact, the complements of congruent angles are always congruent, and, likewise, the supplements of congruent angles are always congruent.

Example:

Each of the following statements is true EXCEPT:

- (A) Vertical angles could be supplementary.
- (B) Vertical angles are never adjacent.
- (C) If $m\angle A = 60^\circ$, $m\angle B = 100^\circ$, and $m\angle C = 20^\circ$, then the angles are supplementary.
- (D) Vertical angles could be complementary.
- (E) The angles of a linear pair are always adjacent.

If the opposite angles formed by two intersecting lines both equal 90° , then A is true. Likewise, if the opposite angles formed by two intersecting lines both equal 45° , then D is true. Vertical angles are never adjacent; they're opposite angles. Linear pairs of angles always share a common side and are therefore adjacent, so B and E are true. On first glance, C may appear to be true, since $60 + 100 + 20 = 180$. Supplementary angles must be a *pair*, however, so C is, in fact, false.

The correct answer is C. Answer

Triangles

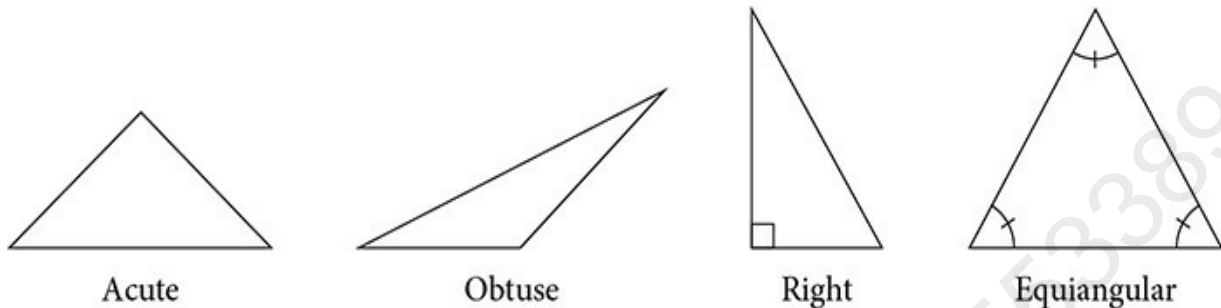
Types of Triangles

The triangle is the most common geometric figure used. A *triangle* is a polygon with three sides. It is formed by connecting three noncollinear points.

Triangles can be classified by their angles:

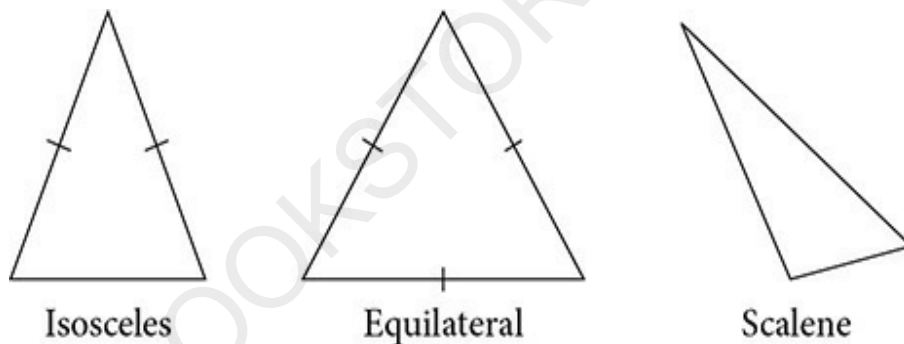
- **Acute.** A triangle with three acute angles

- **Obtuse.** A triangle with one obtuse angle
- **Right.** A triangle with one right angle
- **Equiangular.** A triangle with all angles congruent



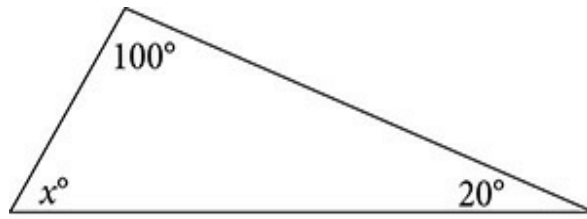
Triangles can also be classified by their sides:

- **Isosceles.** A triangle with at least two congruent sides
- **Equilateral.** A triangle with all sides congruent
- **Scalene.** A triangle with no congruent sides



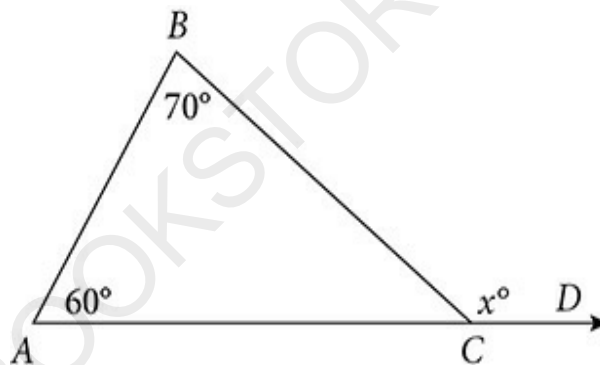
Sum of Interior Angles and Exterior Angles

The *sum of the measures of the interior angles* of a triangle is always 180° . This is a useful theorem that is often used when solving problems involving triangles and other polygons. One way to show this concept is to draw a triangle, cut it out, and tear off each of the three angles. Now, place each angle so that it shares a side with the one next to it. The three angles of the triangle will always form a straight line.



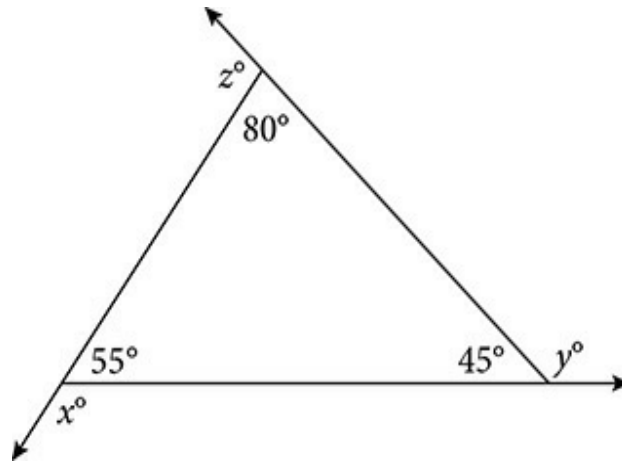
In the figure above, $x + 100 + 20 = 180$, so $x = 60^\circ$. Whenever you know the measures of two angles in a triangle, subtract their sum from 180° to find the measure of the missing angle. A triangle with all angles congruent is called *equiangular*. Each angle of an equiangular triangle measures 60° , since $x + x + x$ must equal 180° . Equiangular triangles are also always equilateral.

An exterior angle is an angle on the outside of a triangle formed by extending one of the triangle's sides. Each exterior angle of a triangle has two remote interior angles and an adjacent interior angle. The *remote interior angles* are the two angles inside the triangle that do not share a vertex with the exterior angle. The *measure of an exterior angle* is equal to the sum of the measures of its two remote interior angles.



In the figure above, $\angle CAB$ and $\angle ABC$ are the remote interior angles for the exterior angle measuring x° . Therefore, $x = 60 + 70 = 130^\circ$. You can check your answer by looking at the *adjacent interior angle* for $\angle BCD$. $\angle BCD$ and $\angle ACB$ are a linear pair, so $m\angle ACB = 50^\circ$. The sum of the interior angles is $50 + 60 + 70$, which does, in fact, equal 180° .

The *sum of the measures of the exterior angles of a triangle* is always 360° . This sum is found by including *one* exterior angle for each vertex of the triangle, not two. In fact, the sum of the measures of the exterior angles of *any* polygon is 360° .



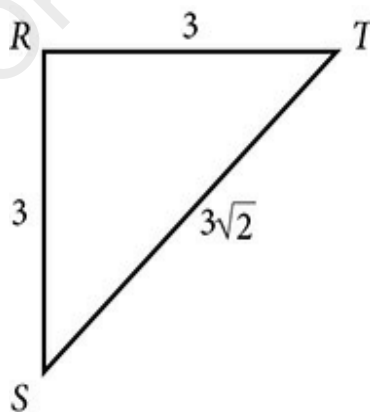
In the figure above, $x + y + z = 360^\circ$. Solve for each exterior angle by subtracting the measure of its adjacent interior angle from 180° .

$$x = 125^\circ, y = 135^\circ, \text{ and } z = 100^\circ$$

$$125 + 135 + 100 = 360^\circ$$

Example:

Find the measure of $\angle SRT$ given $m\angle RTS = m\angle RST = 45^\circ$. Then classify the triangle by its angles and sides.



Let $x = m\angle SRT$.

$$x + 45 + 45 = 180$$

$$x = 90^\circ$$

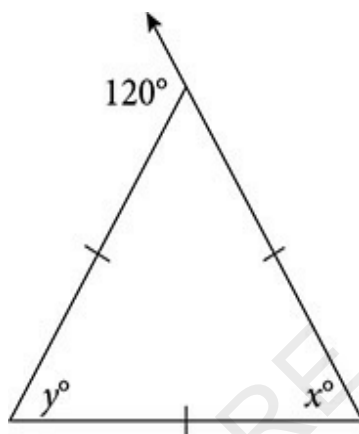
The triangle has one 90° angle, making it a right triangle.

Since the triangle has two congruent sides that measure 3, it is isosceles.

90° ; right, isosceles triangle Answer

Example:

Find the measure of the missing angles. Then classify the triangle by its angles and sides.



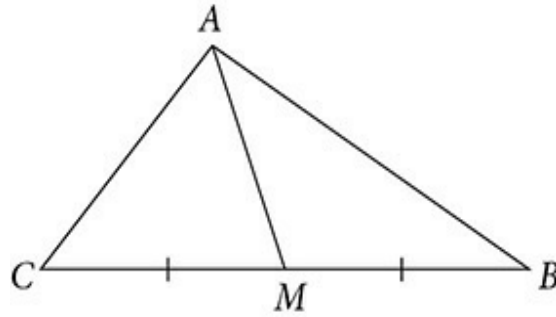
Since the sum of the measures of the two remote interior angles equals the measure of the exterior angle, $x + y = 120$. Notice that the triangle is equilateral. By definition, all equilateral triangles are equiangular. Each angle measures 60° .

$x = 60^\circ$ and $y = 60^\circ$; equiangular, equilateral triangle Answer

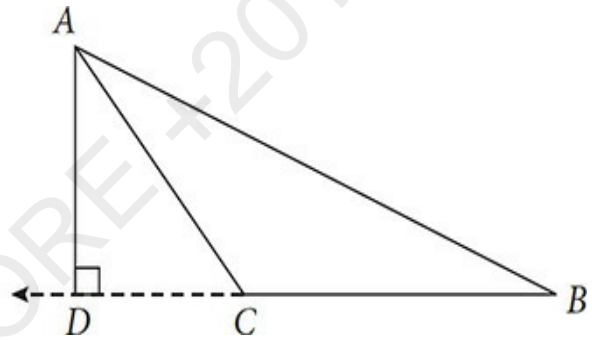
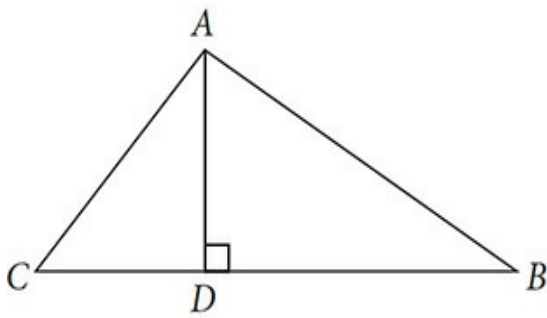
Medians, Altitudes, and Angle Bisectors

A *median* of a triangle is a segment extending from one vertex to the midpoint of the opposite side. Every triangle actually has three medians.

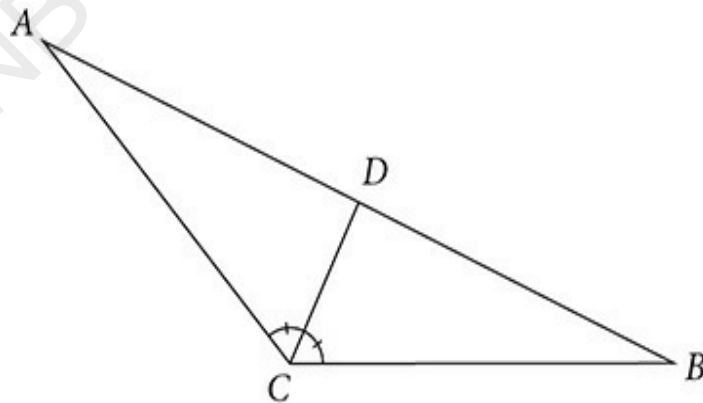
\overline{AM} is a median of $\triangle ABC$.



An *altitude* of a triangle is a segment extending from one vertex and is perpendicular to the opposite side (or the line containing the opposite side). Every triangle also has three altitudes. \overline{AD} is an altitude of each of the triangles below.



An *angle bisector* of a triangle is a segment that divides an interior angle of the triangle into two congruent angles and has an endpoint on the opposite side of the triangle. Every triangle has three angle bisectors. \overline{CD} is a bisector of $\angle C$ in $\triangle ABC$.



The angle bisector of the vertex angle of an isosceles triangle is also a median to the base and an altitude of the triangle.

Example:

Given $\triangle RST$ has an exterior altitude, the triangle could be which of the following?

- (A) Acute
- (B) Obtuse
- (C) Right
- (D) Equilateral

Exterior altitudes extend from one vertex and are perpendicular to *the line containing* the opposite side. This only occurs when a triangle has an obtuse angle.

B; obtuse is the correct answer. Answer

Example:

Given $\triangle MNL$, the measure of $\angle NMP$ is 40° . \overline{MP} bisects $\angle M$, and $\angle N$ is congruent to $\angle LMP$. Find $m\angle L$.

Sketching a diagram of $\triangle MNL$ will help to solve for the measure of $\angle L$. Since $\angle M$ is bisected, the measures of both $\angle NMP$ and $\angle LMP$ are 40° . The measure of $\angle N$ is also 40° , since it is congruent to $\angle LMP$. The measure of $\angle M$ is, therefore, 80° . $m\angle M + m\angle N + m\angle L = 180^\circ$, so $80^\circ + 40^\circ + m\angle L = 180^\circ$.

$m\angle L = 60^\circ$. Answer

Congruent Triangles

Triangles are congruent if they have the same shape and size. In other words, given two congruent triangles, if you cut out one and place it over the other, they will perfectly overlap. *Congruent triangles* have three pairs of congruent sides and three pairs of congruent angles.

SSS, SAS, and ASA Postulates and the AAS Theorem

It is not necessary, however, to compare all six parts of two triangles (three angles and three sides) to prove congruency. Comparing three parts, in some cases, will do. The following four postulates illustrate “shortcuts” to proving triangles congruent.

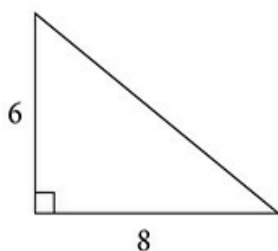
1. **Side Side Side (SSS).** If three sides of one triangle are congruent to three sides of another, then the triangles are congruent.
2. **Side Angle Side (SAS).** If two sides and the included angle of one triangle are congruent to two sides and the included angle of another, then the triangles are congruent.
3. **Angle Side Angle (ASA).** If two angles and the included side of one triangle are congruent to two angles and the included side of another, then the triangles are congruent.
4. **Angle Angle Side (AAS).** If two angles and the side not between them of one triangle are congruent to two angles and the side not between them of another, then the triangles are congruent.

An *included angle* is the angle between the given pair of sides. Likewise, an *included side* is the side between the given pair of angles. Notice that SSA and AAA are not included in the above list. Knowing two sides and an angle not between them are congruent or knowing three angles are congruent does not prove triangle congruency. Take two equilateral triangles. Equilateral triangles must also be equiangular, so each angle measures 60° . Say the first triangle has sides measuring 1 cm and the second triangle has sides measuring 10 cm. The second triangle is significantly larger. The two equilateral triangles are the same shape, but not the same size. AAA does not prove that these two triangles are congruent.

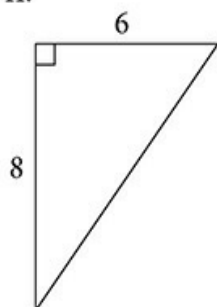
Example:

Which of the following triangles are congruent?

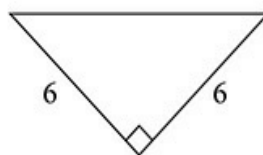
I.



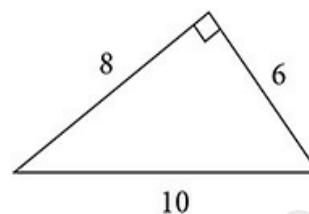
II.



III.



IV.



- (A) I and II
 (B) II and III
 (C) I and IV
 (D) III and IV
 (E) All four are congruent.

Look for SSS, SAS, ASA, or AAS to be true for two triangles. You can immediately eliminate ASA and AAS because you do not know the measure of two angles in any triangle. That leaves SSS and SAS triangles. Triangle I has a side measuring 6, a side measuring 8, and a 90° included angle. Triangle IV has the same, so I and IV are congruent. (Using the Pythagorean Theorem, you can find that the third side of I also measures 10, so SSS also proves I and IV are congruent.)

The correct answer is C. Answer

CPCTC (Corresponding Parts of Congruent Triangles are Congruent)

If triangles are congruent, then the corresponding parts of the triangles are congruent. *CPCTC* stands for *Corresponding Parts of Congruent Triangles are Congruent*. If $\triangle ABC \cong \triangle DEF$, six pairs of corresponding parts are congruent:

$$\overline{AB} \cong \overline{DE} \quad \angle A \cong \angle D$$

$$\overline{BC} \cong \overline{EF} \quad \angle B \cong \angle E$$

$$\overline{AC} \cong \overline{DF} \quad \angle C \cong \angle F$$

Example:

$\triangle JHK$ is congruent to $\triangle JHI$ and the two triangles share \overline{JH} . $\angle K$ is congruent to $\angle I$ and $\angle JHK$ is congruent to $\angle JHI$. Is \overline{HK} congruent to \overline{HI} ?

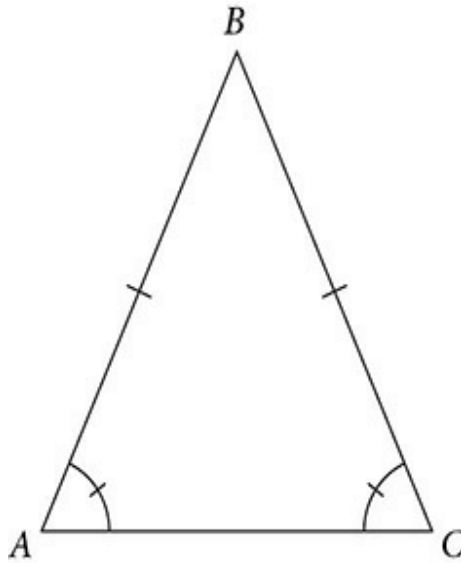
Sketching the two triangles may help answer this question. Determine if the triangles are congruent and then use CPCTC to show the sides are congruent. Two angles and the side not between them (\overline{JH}) are congruent in the two triangles, making the triangles congruent by AAS. Since corresponding parts of the triangles are therefore congruent, you can conclude that $\overline{HK} \cong \overline{HI}$.

Isosceles Triangles

Isosceles triangles have *at least* two sides congruent. By this definition, equilateral triangles are also classified as isosceles. In an *isosceles triangle*, the two congruent sides are the *legs*, the remaining side is the *base*, the *vertex angle* is the angle included by the two congruent sides, and the *base angles* are the angles having the base as a side.

Isosceles Triangle Theorem

The *Isosceles Triangle Theorem* states that if two sides of a triangle are congruent, then the angles opposite the congruent sides (the base angles) are congruent. Given $\triangle ABC$ is isosceles and $\overline{AB} = \overline{BC}$, then $\angle BAC \cong \angle BCA$.



Example:

Which of the following could be the degree measures of the angles of an isosceles triangle?

- I. 40, 40, 80
- II. 45, 45, 90
- III. 30, 60, 90

- (A) I only
- (B) II only
- (C) III only
- (D) I and II only
- (E) II and III only

At first glance, it may seem that both I and II are possible angles of an isosceles triangle. By the Isosceles Triangle Theorem, you know that two of the angles must be equal in measure. The angle measures in I, however, add up to 160° , which is not possible for a triangle. (The three angles must add up to 180° .)

B; II only is the correct answer. Answer

Triangle Inequality

The *Triangle Inequality Theorem* states that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Let's assume $\triangle ABC$ is a right triangle with sides of 3, 4, and 5 cm. Using the Triangle Inequality Theorem, you can show that the following three inequalities are true:

$$3 + 4 > 5$$

$$3 + 5 > 4$$

$$4 + 5 > 3$$

If just one of the inequalities proves not to be true, it is impossible for a triangle to have sides of the given measures. Take a triangle of sides measuring 3, 5, and 8.

$$3 + 8 > 5$$

$$5 + 8 > 3$$

$$3 + 5 > 8 \quad \text{Not true}$$

The third inequality is not valid, meaning that it is impossible to have a triangle with sides measuring 3, 5, and 8 units.

Example:

Which sets of numbers could be the lengths of sides of a triangle?

I. 7, 24, 25

II. 7, 7, 7

III. 1, 2, 3

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) II and III only

Answer II is an equilateral triangle. The Triangle Inequality Theorem is valid because $7 + 7 > 7$. Answer I also passes the Triangle Inequality Theorem because $7 + 24 > 25$, $24 + 25 > 7$, and $7 + 25 > 24$. Answer III is not a valid triangle since $1 + 2$ is not greater than 3.

D; I and II only is the correct answer. Answer

Pythagorean Theorem

The *Pythagorean Theorem* shows a special relationship among the sides of a right triangle. Before getting into the theorem, let's review the parts of a right triangle. The two sides adjacent to the right angle are called the *legs*, and the side opposite the right angle is called the *hypotenuse*. The Pythagorean Theorem states that the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse.

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

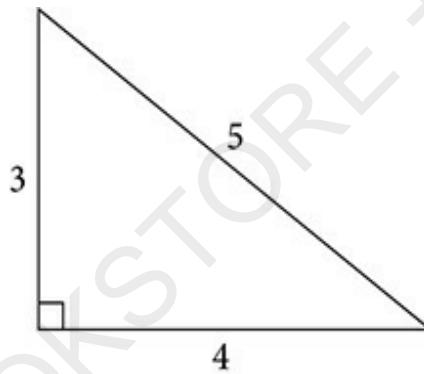
Often, a and b are the variables used to show the lengths of the legs and c is used to show the length of the hypotenuse. The Pythagorean Theorem states that:

$$a^2 + b^2 = c^2$$

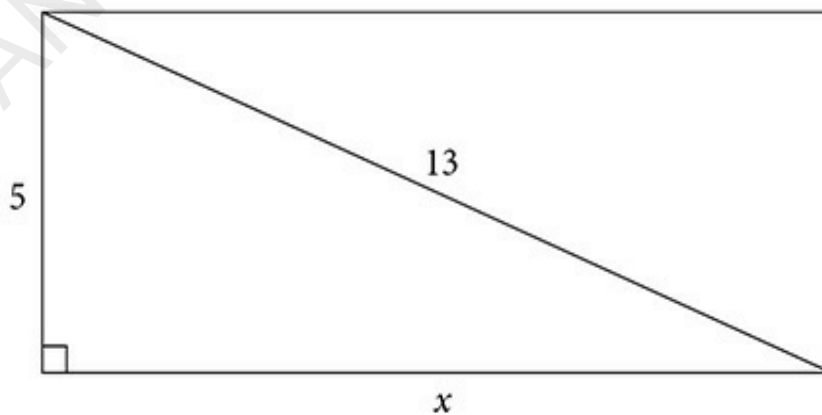
Pythagorean triples are sets of three whole numbers that work in the Pythagorean Theorem. The following table shows a few commonly used triples:

a	b	c
3	4	5
5	12	13
7	24	25
8	15	17

In fact, any multiple of a Pythagorean triple also will work in the Pythagorean Theorem. Take a 3-4-5 right triangle and multiply the sides by 2. 6-8-10 will work in the Pythagorean Theorem. Multiply 3-4-5 by 3 and you'll see that 9-12-15 also works in the Pythagorean Theorem.



Often, the Pythagorean Theorem is applied to word problems involving triangles as parts of other figures. The next example illustrates this concept.



Example:

Find the length of a rectangle whose width is 5 cm and whose diagonal measures 13 cm.

It may be helpful to sketch a diagram to show the given information in the problem. Let x represent the length of the rectangle.

$$5^2 + x^2 = 13^2$$

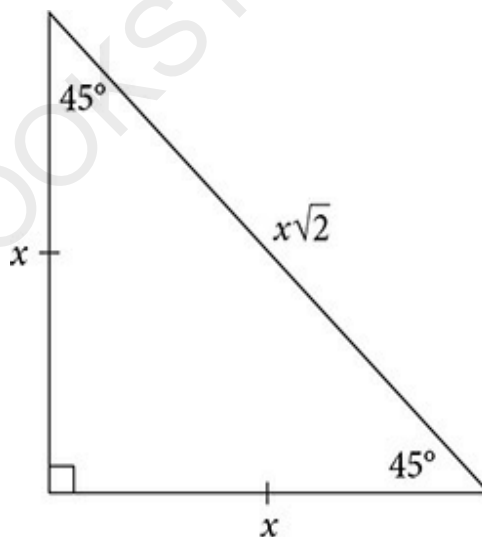
$$x^2 = 13^2 - 5^2$$

$$x^2 = 169 - 25 = 144$$

$$x = 12 \text{ cm} \quad \text{Answer}$$

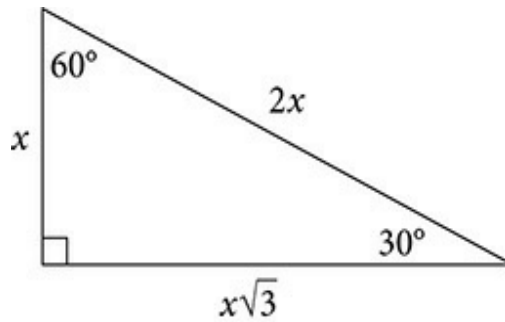
Special Right Triangles

Two special right triangles, a 45° - 45° - 90° and a 30° - 60° - 90° , occur often in mathematics and it is important to understand the relationships between their sides and angles. A 45° - 45° - 90° triangle is an isosceles right triangle whose sides are in the following ratios:



In other words, the length of the hypotenuse is $\sqrt{2}$ times longer than the lengths of its legs.

A 30° - 60° - 90° triangle is a scalene right triangle whose sides are in the following ratios:



The length of the hypotenuse is two times longer than the length of the shorter leg (opposite the 30°) and the length of the longer leg (opposite the 60°) is $\sqrt{3}$ times longer than the length of the shorter leg.

Example:

One leg of a 45° - 45° - 90° triangle is 10 cm long. How long is its hypotenuse?

Using the ratios for the sides of a 45° - 45° - 90° triangle, the hypotenuse is a factor of $\sqrt{2}$ bigger than the legs. Each leg measures 10 cm.

$10\sqrt{2}$ cm Answer

Example:

The hypotenuse of a 30° - 60° - 90° triangle is 7 in. Find the length of its longer leg.

Using the ratios for the sides of a 30° - 60° - 90° triangle, the shorter leg is half of the hypotenuse, $\frac{7}{2}$ inches. The longer leg is $\sqrt{3}$ times the shorter leg.

$\frac{7\sqrt{3}}{2}$ in. Answer

Example:

Find the length of each leg of an isosceles right triangle whose hypotenuse is 6 cm.

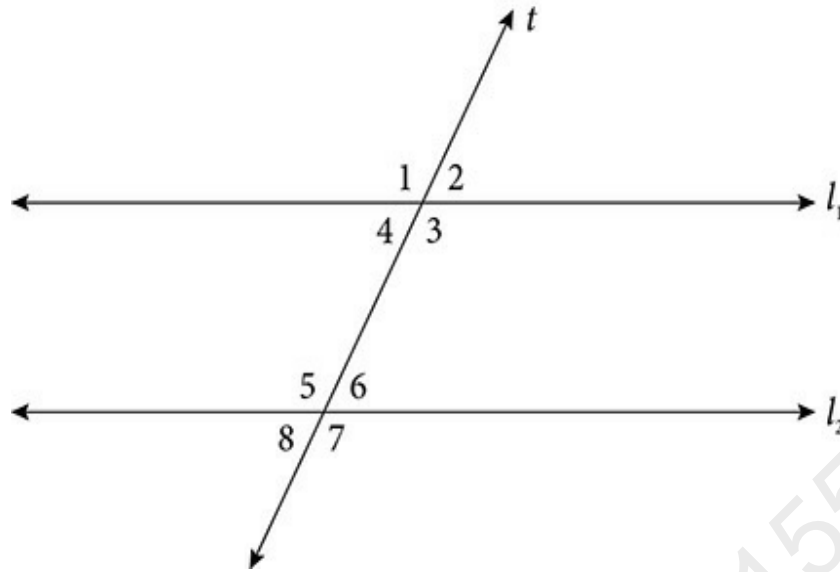
If a right triangle is isosceles, its angles must measure 45° - 45° - 90° . (Remember that the Isosceles Triangle Theorem states that if two sides are congruent, the angles opposite them are also congruent.) The hypotenuse is a factor of $\sqrt{2}$ bigger than the legs. To find the measure of the legs, *divide* 6 cm by $\sqrt{2}$. $\frac{6}{\sqrt{2}}$ is not in simplest form, though. Now, multiply the numerator and denominator by $\sqrt{2}$ to rationalize the denominator.

$$\frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2} \quad \text{Answer}$$

Parallel Lines

Lines that do not intersect and that lie in the same plane are called *parallel lines*. Lines that do not intersect and lie in different planes are called *skew*. Planes, segments, and rays can also be parallel.

The expression $l_1 \parallel l_2$ means “line 1 is parallel to line 2.” Parallel lines are marked by arrows, “>,” in diagrams. A *transversal* is a line that intersects two or more distinct coplanar lines. In the figure below, line t is a transversal and $l_1 \parallel l_2$.



When two lines are cut by a transversal, special angles are created. Based on the figure above, the angles are as follows:

- **Corresponding angles.** $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$

A pair of nonadjacent angles, one interior and the second exterior, on the same side of the transversal

- **Alternate interior angles.** $\angle 3$ and $\angle 5$, $\angle 4$ and $\angle 6$

A pair of nonadjacent, interior angles on opposite sides of the transversal

- **Alternate exterior angles.** $\angle 1$ and $\angle 7$, $\angle 2$ and $\angle 8$

A pair of nonadjacent, exterior angles on opposite sides of the transversal

- **Interior angles on the same side of the transversal.** $\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$

- **Exterior angles on the same side of the transversal.** $\angle 1$ and $\angle 8$, $\angle 2$ and $\angle 7$

When the two lines cut by the transversal are parallel, then corresponding angles, alternate interior angles, and alternate exterior angles are *congruent*. Interior angles on the same side of the transversal and exterior angles on the same side of the transversal are *supplementary*. If two parallel lines are cut by a transversal and the transversal is perpendicular to one of the parallel lines, you can conclude that it is also perpendicular to the second parallel line. Each of the eight angles formed measures 90° .

Example:

Given $l_1 \parallel l_2$ in the figure on the previous page, $m\angle 1 = 2x$, and $m\angle 2 = x$, find the measure of $\angle 7$.

$\angle 1$ and $\angle 2$ are a linear pair of angles and are, therefore, supplementary.

$$m\angle 1 + m\angle 2 = 180$$

$$2x + x = 180$$

$$3x = 180$$

$$x = 60^\circ$$

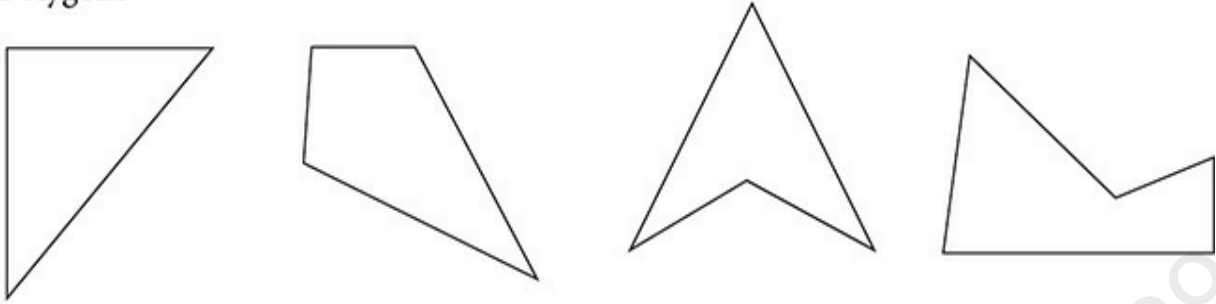
Thus, $m\angle 1 = 2(60) = 120^\circ$. $\angle 1$ and $\angle 7$ are alternate exterior angles, and therefore, they are congruent.

$$m\angle 7 = 120^\circ \quad \text{Answer}$$

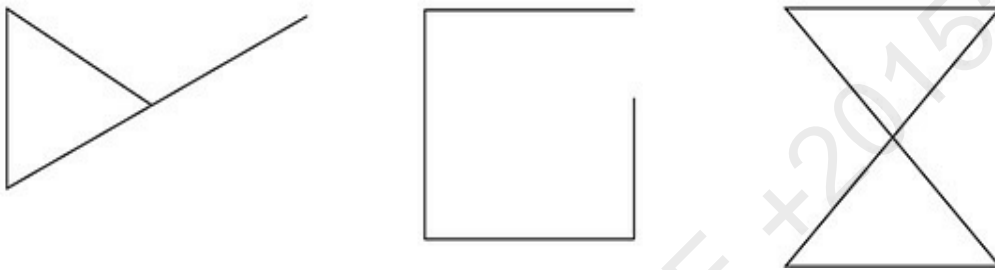
Polygons

A *polygon* is a many-sided closed figure. It consists of line segments connected endpoint to endpoint.

Polygons



Not Polygons



The line segments are called *sides*, and each endpoint is called a *vertex*. (The plural of vertex is “vertices.”) A *diagonal* is a segment that connects one vertex to another, nonconsecutive vertex. (A segment connecting one vertex to another, *consecutive* vertex is a side.) Rectangle $ABCD$, for example, has two diagonals \overline{AC} and \overline{BD} .

A *regular polygon* is both equiangular and equilateral. A square is an example of a regular polygon.

Types of Polygons

Polygons are named according to their number of sides.

NUMBER OF SIDES	NAME OF POLYGON
2	Doesn't exist
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon
11	Hendecagon or undecagon
12	Dodecagon
n	n -gon

Example:

How many diagonals can be drawn from one vertex of a decagon?

Think about the problem. A decagon has 10 vertices. You can connect the given vertex to 7 nonconsecutive vertices, not including the 2 vertices next to the given one, and the given vertex itself. Therefore, 7 diagonals can be drawn. $7 = 10 - 3$. The expression $n - 3$ (where n is the number of sides) represents the number of diagonals from one vertex of any polygon.

The correct answer is 7. Answer

Example:

How many *total* diagonals can you draw in an octagon?

The total number of diagonals can be expressed as $\frac{n(n-3)}{2}$, where n is the number of sides. A rectangle, for example, has two diagonals:

$$\frac{4(4-3)}{2} = 2.$$

An octagon has $\frac{8(8-3)}{2} = \frac{40}{2} = 20$ diagonals.

20 diagonals Answer

Perimeter

The *perimeter* of a polygon is the sum of the lengths of its sides.

Example:

Find the perimeter of a regular pentagon whose sides measure 13 cm.

This is a straightforward perimeter calculation. Since the pentagon is regular, it is equilateral. The perimeter is simply $13 \times 5 = 65$ cm.

65 cm Answer

Example:

A quadrilateral has sides x , $3x$, $4x - 1$, and $2x + 3$. Find the length of the longest side, given the perimeter is 72 cm.

Since the perimeter is the sum of the lengths of all four sides, you know that:

$$x + 3x + 4x - 1 + 2x + 3 = 72$$

$$10x + 2 = 72$$

$$10x = 70$$

$$x = 7$$

The longest side measures $4x - 1$, which is $4(7) - 1$ or 27 cm.

27 cm Answer

Sum of the Interior Angles

The sum of the measures of the angles of a triangle is 180° . The sum of the measures of the angles of a quadrilateral is 360° . The sum of the measures of the angles of a pentagon is 540° . The sum of the measures of the angles of a polygon can, therefore, be expressed as $180(n - 2)^\circ$, where n is the number of sides.

Example:

Find the measure of each angle in a regular decagon.

The sum of all 10 angles in a decagon is $180(10 - 2)$ or $1,440^\circ$. A *regular* decagon is equiangular, meaning that each angle has the same measure. Each angle, therefore, measures $1,440 \div 10$ or 144° .

144° Answer

Example:

Each of the following statements is true EXCEPT:

- (A) The larger the number of sides of a polygon, the greater the sum of its interior angles.
- (B) The sum of the interior angles of a polygon is always a multiple of 180° .
- (C) There is a polygon whose interior angles add up to 900° .

- (D) There is a polygon whose interior angles add up to 800° .
(E) Any interior and exterior angles of a regular polygon are supplementary.

It is possible to have a polygon whose angles add up to 900° , since 900 is a multiple of 180. 800 is not a multiple of 180, however, so it is not possible to have a polygon whose angles add up to 800° .

D is the correct answer. Answer

Sum of the Exterior Angles

The sum of the exterior angles of any polygon is 360° . This assumes that the polygon is convex and that there is one exterior angle at each vertex.

Example:

Find the measure of each exterior angle of a regular hexagon.

The sum of all 6 exterior angles of a hexagon is 360° . A regular hexagon is equiangular, meaning that each interior angle has the same measure. Each exterior angle will also have the same measure since it forms a linear pair with its adjacent interior angle. Each exterior angle, therefore, measures $360 \div 6$ or 60° .

60° Answer

Example:

The measure of each interior angle of a regular polygon is 120° more than its adjacent exterior angle. How many sides does the polygon have?

Let x = the measure of the exterior angle and $120 + x$ = the measure of the interior angle. Since an interior angle of a polygon and its adjacent exterior angle are supplementary, you know that:

$$x + (120 + x) = 180$$

$$2x + 120 = 180$$

$$2x = 60$$

$$x = 30$$

Knowing the sum of the exterior angles must be 360° , solve for the number of sides using:

$$\frac{360}{n} = 30$$

$$\frac{360}{30} = n$$

$$n = 12$$

The polygon has 12 sides. Answer

Special Quadrilaterals

A *quadrilateral* refers to any four-sided polygon. Quadrilaterals with special properties are named as follows:

QUADRILATERAL	DEFINITION
Trapezoid	A quadrilateral with exactly one pair of parallel sides. A trapezoid can also be right or isosceles.
Parallelogram	A quadrilateral with two pairs of parallel sides.
Rhombus	An equilateral parallelogram.
Rectangle	An equiangular parallelogram.
Square	An equilateral and equiangular parallelogram. A square can also be defined as an equilateral rectangle, an equiangular rhombus, or a regular parallelogram.

Notice that rhombi, rectangles, and squares are also types of parallelograms. This means that all of the properties of a parallelogram also apply to these three shapes. A square is unique in that it can be classified as three figures: a parallelogram, a rectangle, and a rhombus.

Example:

Given parallelogram $ABCD$, each of the following statements is true EXCEPT:

- (A) A pair of opposite sides is parallel and congruent.
- (B) Both pairs of opposite angles are congruent.
- (C) The diagonals are perpendicular.
- (D) The diagonals bisect each other.
- (E) All pairs of consecutive angles are supplementary.

Diagonals of a parallelogram do bisect each other, but they are not perpendicular bisectors. (The diagonals of a rhombus and a square are perpendicular bisectors, though.)

C is the correct answer. Answer

The four true answers in the example, A, B, D, and E, are important characteristics of parallelograms. Two more characteristics can be added to this list:

1. Both pairs of opposite sides are parallel.
2. Both pairs of opposite sides are congruent.

If one of these six characteristics is true for a quadrilateral, you can conclude that the quadrilateral is a parallelogram.

Similarity

Similar figures have the same shape but different size. A square with sides of 2 cm is *similar* to a square with sides of 4 cm. Both have the same shape, but the second square is twice the size of the first. Note the difference between

congruent figures and similar figures. Congruent figures have both the same shape and the same size.

Ratio and Proportion

The *ratio* of a to b is the quotient $\frac{a}{b}$, where $b \neq 0$. Ratios can be written as “ a to b ,” $\frac{a}{b}$, $a:b$. A ratio is in simplest form when it is impossible to divide the terms by a common factor bigger than 1.

Example:

If the ratio of the lengths of segment \overline{AB} to segment \overline{CD} is 3:5 and $\overline{AB} = 15$ cm, find the measure of segment \overline{CD} .

Let $y = \overline{CD}$. $\frac{15}{y} = \frac{3}{5}$. Cross multiply to solve for y , resulting in:

$$3y = 75$$

$$y = 25 \text{ cm} \quad \text{Answer}$$

You may also have immediately seen that 15 is 5 times larger than 3, so $5(3) = 15$.

We used a proportion to solve the previous problem. A *proportion* is an equation in which both sides are ratios. Proportions can be solved using cross multiplication:

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = cb.$$

Example:

The measures of the angles of a triangle are in the ratio of 2:3:4. Find the measure of the smallest angle.

You know that the sum of the angles in a triangle is 180° so:

$$2x + 3x + 4x = 180$$

$$9x = 180$$

$$x = 20$$

The smallest angle therefore measures $2(20) = 40^\circ$.

40° Answer

Example:

Write the ratio $\frac{2x^2 - 18}{x - 3}$ in simplest form.

First, distribute out a common factor of 2 in the numerator.

$$\frac{2(x^2 - 9)}{x - 3}$$

The numerator is the difference of perfect squares, so it can be further factored to:

$$\frac{2(x - 3)(x + 3)}{x - 3}$$

Dividing the numerator and denominator by a factor of $x - 3$ gives you:

$$\frac{2(x + 3)}{1} \quad \text{Answer}$$

Similar Triangles

We've previously mentioned that similar figures have the same shape but

different size. Specifically, two polygons are *similar* if their corresponding angles are congruent and their corresponding sides have proportional lengths. The statement $\triangle ABC \sim \triangle DEF$ reads, “triangle ABC is similar to triangle DEF .”

Example:

Each of the following statements is true EXCEPT:

- (A) Any two equilateral triangles are similar.
- (B) Any two isosceles triangles are similar.
- (C) Any two congruent polygons are similar.
- (D) Any two squares are similar.
- (E) Any two regular pentagons are similar.

Remember that similar figures must have the same shape. Isosceles triangles may or may not have the same shape. They are similar only when the vertex angles (the angle included by the two congruent sides) are congruent.

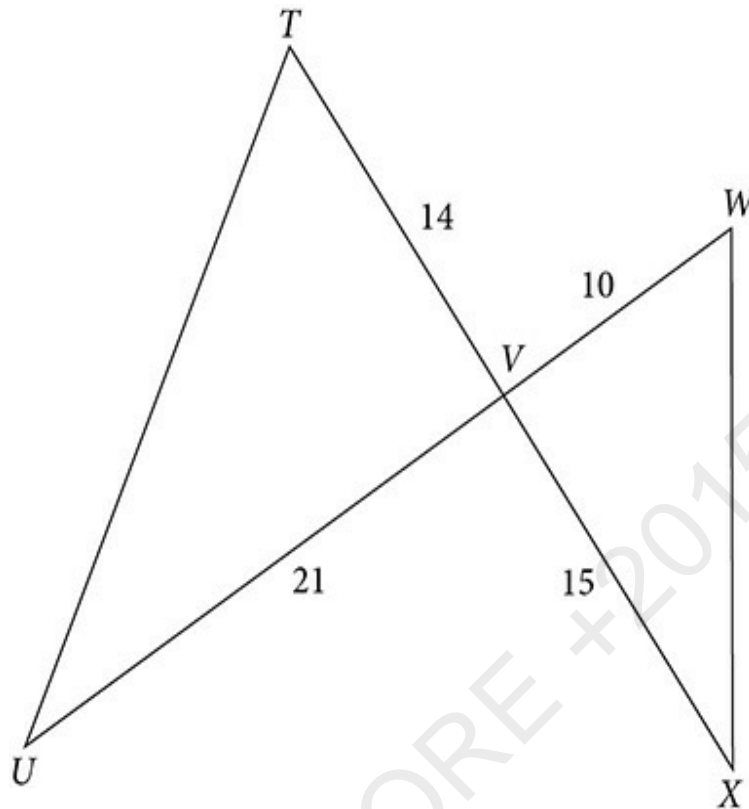
B is the correct answer. Answer

When you are proving two triangles similar, it is not necessary to compare all three pairs of corresponding angles and all three pairs of corresponding sides. There are “shortcuts” to proving triangles similar just as there are “shortcut” postulates, such as SAS and ASA, to prove triangles congruent.

1. **Angle Angle (AA).** If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
2. **Side Angle Side (SAS).** If an angle of one triangle is congruent to an angle of another triangle and the sides including those angles are proportional, then the triangles are similar.
3. **Side Side Side (SSS).** If the corresponding sides of two triangles are proportional, then the triangles are similar.

Example:

Is $\Delta TUV \sim \Delta WXV$?



$\angle TVU$ and $\angle WVX$ are vertical angles and are, therefore, congruent. Now, take a look at the ratio of the lengths of corresponding sides:

$$\frac{\overline{WV}}{\overline{TV}} = \frac{10}{14} = \frac{5}{7}$$

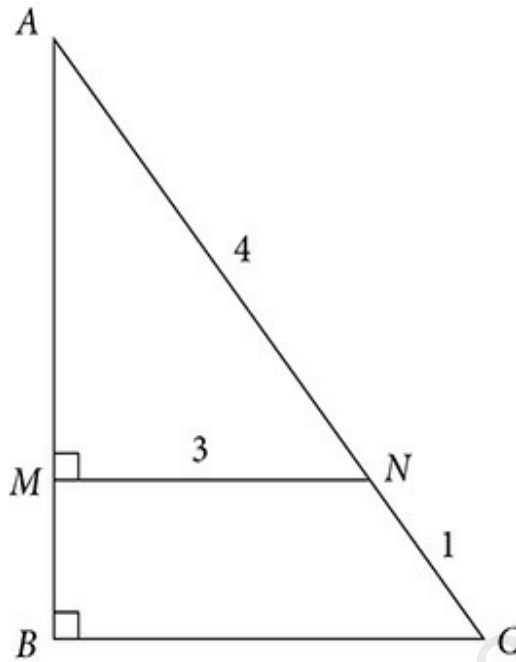
$$\frac{\overline{XV}}{\overline{UV}} = \frac{15}{21} = \frac{5}{7}$$

By the Side Angle Side theorem, SAS, the two triangles are similar.

Yes, $\Delta TUV \sim \Delta WXV$. Answer

Example:

Given $\Delta AMN \sim \Delta ABC$, find \overline{BC} .



Note that $\triangle AMN \sim \triangle ABC$ by the AA theorem. $\angle AMN$ and $\angle ABC$ are both 90° and both triangles share a common angle, $\angle A$. Because the triangles are similar, the sides must be proportional.

$$\frac{\overline{AC}}{\overline{AN}} = \frac{5}{4}$$

$$\frac{\overline{BC}}{\overline{MN}} = \frac{\overline{BC}}{3}$$

$$\frac{\overline{BC}}{3} = \frac{5}{4} \text{ so } \overline{BC} \text{ must measure } \frac{15}{4} \text{ units.}$$

$$\overline{BC} = \frac{15}{4} \quad \text{Answer}$$

The *Triangle Proportionality Theorem* actually states that if a line is parallel to one side of a triangle and intersects the other two sides, then the line divides those sides proportionally. The converse of this theorem is also true. If a line divides two sides of a triangle proportionally, then it is parallel to the third side. The converse was true in the previous example, $\overline{MN} \parallel \overline{BC}$.

Circles

A *circle* is the set of all points in a plane at a given distance from a given point. The given point is called the *center* and the given distance is called the *radius*. All radii in a circle must, therefore, be congruent. Circles with congruent radii are congruent.

Chords

A *chord* is a segment whose endpoints are on the circle. A chord that passes through the center of the circle is called the *diameter*. The diameter is the longest possible chord of a circle and measures twice the length of the radius. A *secant* is a line, ray, or segment containing a chord. Two properties of chords are as follows:

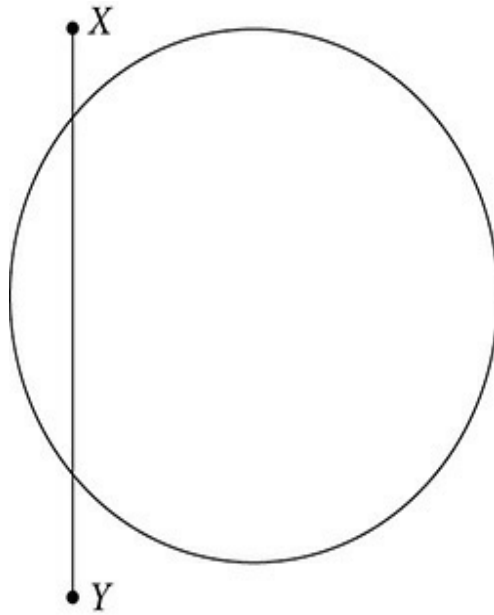
- A segment from the center of a circle perpendicular to a chord bisects the chord.
- Two chords are congruent if they are equally distant from the center of the circle.

Example:

Each of the following statements is true EXCEPT:

- (A) A chord of a circle could also be a diameter.
- (B) A radius of a circle can never be a chord.
- (C) If two circles are congruent, then their diameters are congruent.
- (D) Concentric circles share the same center.
- (E) If \overline{XY} is a secant of a circle, then \overline{XY} is also a chord of the circle.

By definition, answers A, B, C, and D are true statements. E could be true, but a counterexample to show that it is false is as shown below:

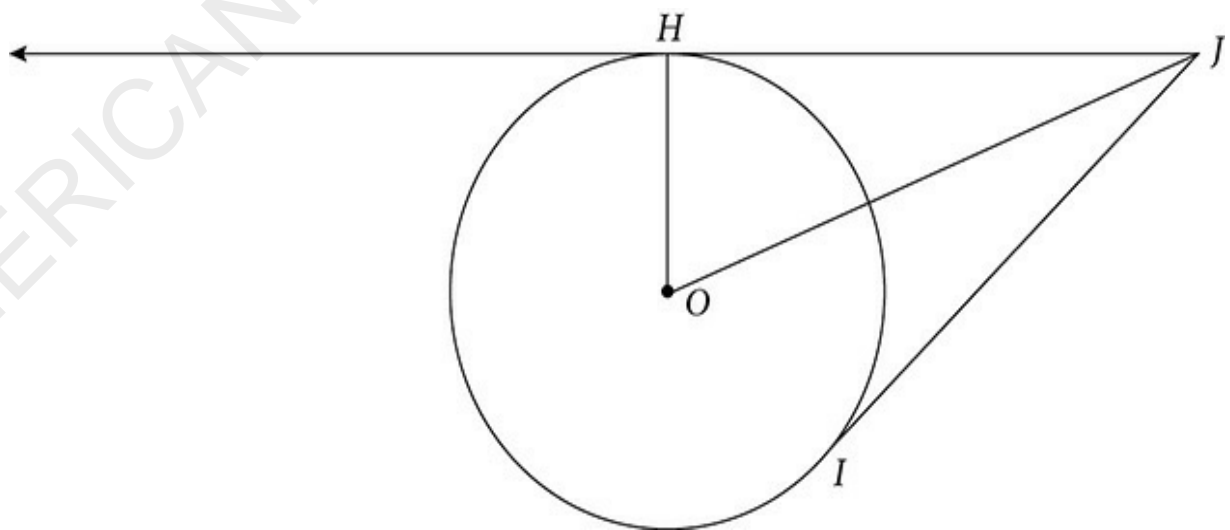


Although \overline{XY} is a secant, the points X and Y do not lie on the circumference of the circle. \overline{XY} is not a chord.

E is the correct answer. Answer

Tangents

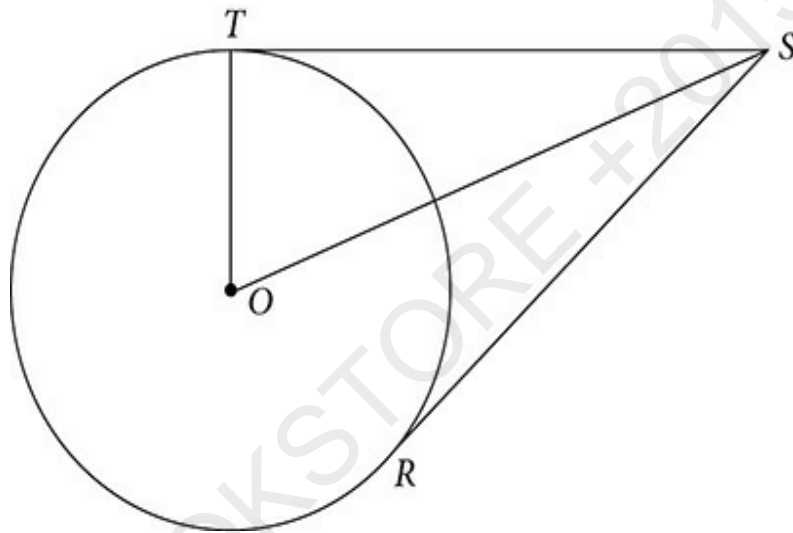
A *tangent* to a circle is a line that intersects a circle at exactly *one* point and that lies in the plane of the circle. At the point of tangency, the tangent forms a 90° angle with the radius. In the following figure, ray \overline{JH} is a tangent and is perpendicular to radius \overline{OH} .



\overline{JH} and \overline{JI} in the figure above are called tangent segments. *Tangent segments* from a given exterior point (in this case, point J) to the circle are congruent. The ray from the exterior point J through the center O of the circle bisects $\angle HJI$, the angle formed by the tangent segments.

Example:

In the figure below, \overline{RS} and \overline{TS} are tangent segments. $\overline{SO} = 10$, $\overline{ST} = 8$, and $m\angle TOS = 53^\circ$. Find \overline{OT} , \overline{RS} , and $m\angle RSO$.



$\overline{ST} = \overline{RS}$ since both are tangent segments from the same exterior point S . \overline{RS} must also measure 8 units. A tangent segment is perpendicular to the radius at the point of tangency, making $m\angle OTS = 90^\circ$. This makes $\triangle OTS$ a right triangle. Use the Pythagorean Theorem to find the length of \overline{OT} .

$$\overline{ST}^2 + \overline{OT}^2 = \overline{OS}^2$$

$$8^2 + \overline{OT}^2 = 10^2$$

$$\overline{OT}^2 = 100 - 64 = 36$$

$$\overline{OT} = 6$$

\overline{SO} bisects $\angle RST$. Since you know two angles in $\triangle OTS$, the third angle, $\angle OST$, equals $180 - (90 + 53) = 37^\circ$. $\angle RSO$ and $\angle OST$ are congruent, so

$$m\angle RSO = 37^\circ.$$

$$OT = 6, RS = 8, \text{ and } m\angle RSO = 37^\circ. \quad \text{Answer}$$

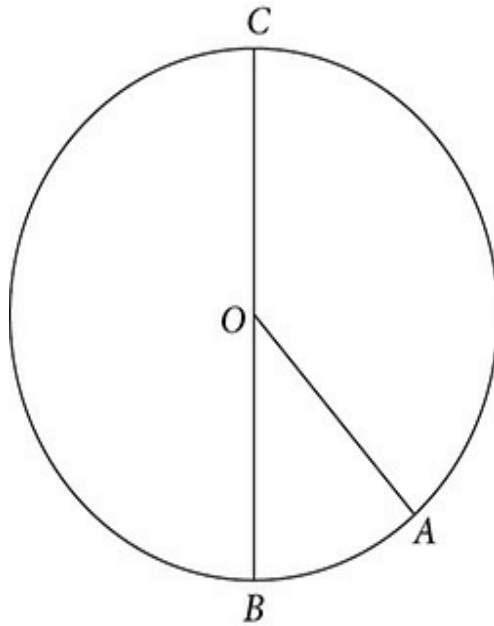
Arcs and Angles

Circles can be broken into three types of arcs:

1. **Semicircle.** An arc that represents half of the circle. A semicircle measures 180° and is named using three letters (e.g., \widehat{ABC}).
2. **Minor arc.** An arc that measures less than 180° . Two letters are used to name a minor arc (e.g., \widehat{AB}).
3. **Major arc.** An arc that measures greater than 180° . Three letters are used to name a major arc (e.g., \widehat{ABC}).

Arc measure is always given in degrees. Remember, a circle measures 360° , so the measure of an arc must be between 0 and 360° . Arc measure is different than “arc length,” so be careful not to confuse the two terms.

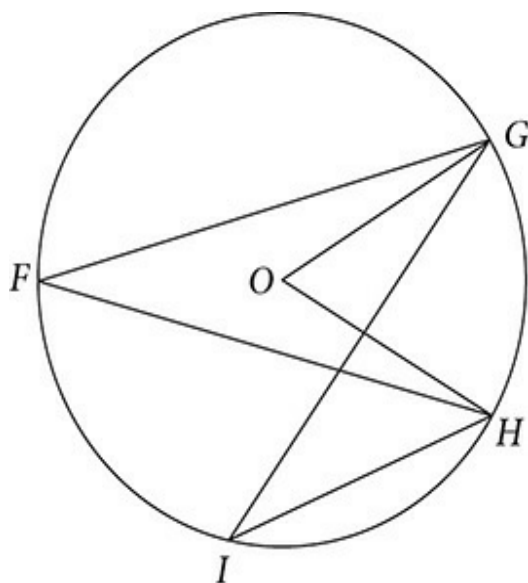
A *central angle* is an angle of a circle whose vertex is the center of the circle. (In the figure below, $\angle AOB$ and $\angle COA$ are central angles.) The *measure of a minor arc* equals the measure of its central angle. Thus, the measure of \widehat{AB} equals the measure of $\angle AOB$. Let $m\angle AOB = 100^\circ$; then $m\widehat{AB} = 100^\circ$. It is also important to note that:



- Two minor arcs are congruent when their central angles are congruent.
- Two minor arcs are congruent when their chords are congruent.
- Two chords are congruent if their corresponding minor arcs are congruent.

The *measure of a major arc* equals 360° minus the measure of its corresponding minor arc. In the figure above, $m \widehat{ABC} = 360 - m \widehat{AC}$. The *measure of a semicircle* always equals 180° .

An *inscribed angle* is an angle of a circle whose vertex is on the circle and whose sides include chords of the circle. The *measure of an inscribed angle* equals half the measure of its intercepted arc. In the following figure, $\angle GFH$ and $\angle GIH$ are inscribed angles.



Example:

In the figure above, $m\angle GOH = 60^\circ$. Find $m\widehat{GH}$, $m\angle GFH$, and $m\angle GIH$.

\widehat{GH} measures 60° , since it equals the measure of its central angle.

$\angle GFH$ and $\angle GIH$ are inscribed angles, so they measure half of their intercepted \widehat{GH} . Both therefore measure 30° .

$$m\widehat{GH} = 60^\circ, m\angle GFH = 30^\circ, m\angle GIH = 30^\circ \quad \text{Answer}$$

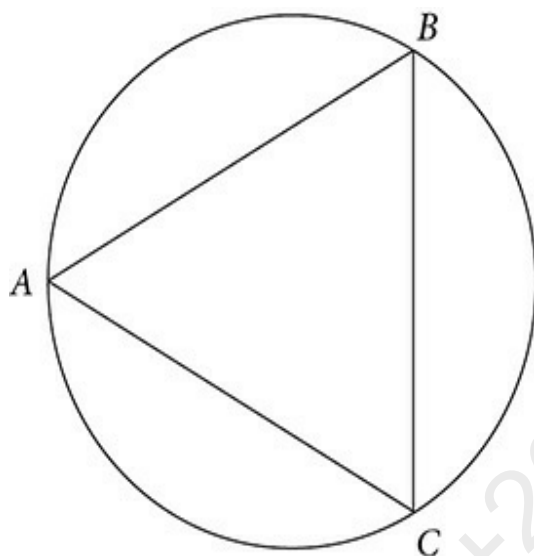
Example:

Equilateral $\triangle ABC$ is inscribed in a circle. Each of the following statements is true EXCEPT:

- (A) $m\widehat{BC} = 60^\circ$.
- (B) $m \text{ minor } \widehat{AB} = m \text{ minor } \widehat{BC}$.
- (C) $m \text{ major } \widehat{CAB} = m \text{ major } \widehat{BCA}$.
- (D) AC — and AB — are equidistant from the center of the circle.

(E) $m\angle ABC = 60^\circ$.

It is helpful to draw a diagram to visualize the given information.



Congruent chords intercept congruent arcs. Therefore, m minor $\widehat{AB} = m$ minor \widehat{BC} , since $AB = BC$. B is true and, likewise, C is true. Because they are congruent chords, AC — and AB — must be equidistant from the center of the circle. D is also true. $\angle ABC$ is an inscribed angle and its measure is 60° , because all angles in an equilateral triangle are 60° . That leaves answer A . \widehat{BC} is intercepted by inscribed angle, $\angle BAC$. Since $m\angle BAC = 60^\circ$, $m\widehat{BC}$ must be $2(60)$ or 120° . A is the correct answer.

An alternative way of solving the problem is to realize that \widehat{AB} , \widehat{BC} , and \widehat{AC} are congruent arcs. Since the whole circle measures 360° , each arc must

measure $\frac{360}{3}$ or 120° . You can immediately deduce that A is not true.

A Answer

Circumference

Circumference is the distance around a circle. It is calculated using one of the following formulas:

$$C = 2\pi r \quad \text{or} \quad C = \pi d$$

where r = the length of the radius and d = the length of the diameter.

Pi, written as the Greek letter π , is the ratio of the circumference of a circle to its diameter. $\pi = \frac{C}{d}$. π is an irrational number, meaning it's nonterminating and nonrepeating. $\pi = 3.14159 \dots$ and it is typically approximated as 3.14, or $\frac{22}{7}$.

Example:

Find the circumference of a circle whose radius is 11 cm. Leave your answer in terms of π .

This problem simply involves substituting values into the circumference formula.

$$C = 2\pi r$$

$$C = 2\pi(11)$$

$$C = 22\pi \text{ cm} \quad \text{Answer}$$

Example:

The radius of one circle is 8 cm more than the radius of a smaller circle. If the ratio of the two circles' circumferences is 3:1, find the circumference of the larger circle.

The circumference of the smaller circle is $2\pi r$, and the circumference of the larger circle is $2\pi(8 + r)$. Set up a proportion to solve for r :

$$\frac{2\pi(8+r)}{2\pi r} = \frac{3}{1}$$

$$\frac{(8+r)}{r} = \frac{3}{1}$$

$$3r = 8 + r$$

$$2r = 8$$

$$r = 4$$

If the radius of the smaller circle is 4 cm so the radius of the larger circle is $4 + 8 = 12$, its circumference is:

$$2\pi(12) = 24\pi$$

24π cm Answer

Arc Length

An *arc length* is a fraction of the circumference of a circle. Arc length is measured in units such as centimeters, inches, feet, and meters, whereas arc measure is measured in degrees. The formula for arc length is as follows:

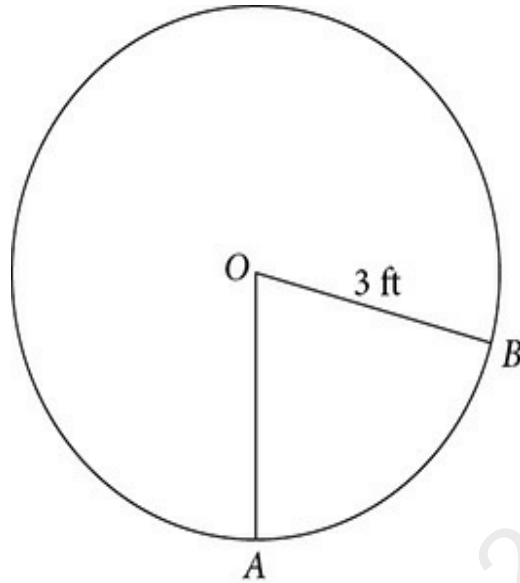
$$l = \frac{x}{360}(2\pi r)$$

where x° = the arc measure.

A semicircle, for example, has an arc length equal to half the circumference, since $\frac{180}{360} = \frac{1}{2}$.

Example:

Find the length of \widehat{AB} given $m\angle AOB = 60^\circ$.



The measure of \widehat{AB} is 60° , since the arc measure equals the measure of the central angle. The length of \widehat{AB} is therefore:

$$l = \frac{60}{360} [2\pi(3)]$$

$$l = \frac{1}{6} (6\pi)$$

π ft Answer

Example:

$\angle EFD$ is inscribed in a circle and measures 70° . If the length of major \widehat{EFD} is 33π inches, find the diameter of the circle.

Since $\angle EFD$ is an inscribed angle, the arc it intercepts, \widehat{ED} , measures $2(70)$ or 140° . m major $\widehat{EFD} = 360 - 140 = 220^\circ$. Write an equation for the length of \widehat{EFD} and solve for r .

$$l = \frac{x}{360}(2\pi r)$$

$$33\pi = \frac{220}{360}(2\pi r)$$

$$33 = \frac{11}{18}(2r)$$

$$33 = \frac{11}{9}(r)$$

$$\frac{33(9)}{11} = r$$

$$27 \text{ inches} = r$$

The diameter is $2(27)$ or 54 inches. Answer

Area

Area Formulas

Area is the measure of the region enclosed by a figure. Every polygon has a unique area, and congruent polygons have equal areas. Area is measured in *square units*, such as centimeters², inches², feet², and meters². Most of the formulas used for calculating area require a base and height. Recall that the height of a polygon is the length of its altitude. The base, however, can be any side of the polygon (not just the “bottom” side as some students believe). The height must measure the altitude that’s perpendicular to the *given* base.

Common area formulas are as follows:

SHAPE	AREA FORMULA	VARIABLE DEFINITIONS
Rectangle	$A = bh$	b = the length of the base h = the height, the length of the altitude
Triangle	$A = \frac{1}{2}bh$	b = the length of the base h = the height, the length of the altitude
Right triangle	$A = \frac{1}{2}\text{leg}_1 \times \text{leg}_2$	leg_1 and leg_2 are the legs (the sides adjacent to the right angle)
Equilateral triangle	$A = \frac{s^2\sqrt{3}}{4}$	s = the length of a side
Square	$A = s^2$	s = the length of a side
Parallelogram	$A = bh$	b = the length of the base h = the height, the length of the altitude
Rhombus	$A = bh$ or $A = \frac{1}{2}d_1d_2$	b = the length of the base h = the height, the length of the altitude d_1 and d_2 = the length of diagonals 1 and 2
Trapezoid	$A = \frac{1}{2}(b_1 + b_2)h$	b_1 and b_2 = the length of the 2 bases, i.e., the parallel sides h = the height, the length of the altitude
Regular polygon	$A = \frac{1}{2}asn$ or $A = \frac{1}{2}ap$	a = the length of the apothem s = the length of a side n = the number of sides p = the perimeter
Circle	$A = \pi r^2$	r = the length of the radius
Sector	$A = \frac{x}{360}\pi r^2$	r = the length of the radius x = the arc measure

The shapes above should be familiar to you. A *sector* is a part of a circle

that resembles a “slice” of the circle. Its edges are two radii and an arc.

Example:

Find the area of a circle with a circumference of 108π cm.

Since $C = 2\pi r$, set up an equation for circumference and solve for r .

$$C = 2\pi r$$

$$108\pi = 2\pi r$$

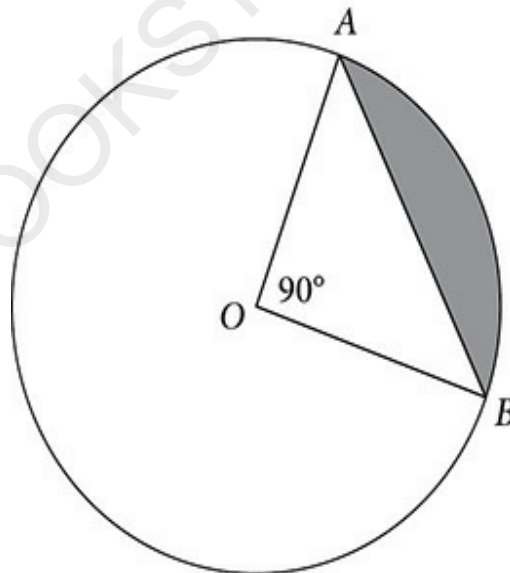
$$54 = r$$

The area formula for a circle is $A = \pi r^2$, so $A = \pi(54)^2$.

$$2,916\pi \text{ cm}^2 \quad \text{Answer}$$

Example:

Find the area of the shaded region given that the radius is 12 m.



Start by looking at the area of the sector, the “slice” of the circle, which includes the shaded region. The area of the sector is as follows:

$$A = \frac{90}{360} \pi(12^2)$$

$$A = \frac{1}{4} \pi(144)$$

$$A = 36\pi \text{ cm}^2$$

The problem is asking only for the area of the shaded region, not the area of the entire sector. Subtract the area of the right triangle $\triangle AOB$ from the area of the sector. Note that the right triangle is an isosceles right triangle whose both legs are formed by radii of the circle.

Area of the sector – Area of the triangle

$$36\pi - \frac{1}{2}bh$$

$$36\pi - \frac{1}{2}(12)(12)$$

$$36\pi - 72 \text{ cm}^2 \quad \text{Answer}$$

Example:

Find the area of a regular pentagon with a perimeter of 75 in and an apothem of 6 in.

Since the pentagon is *regular*, you can apply the area formula. This formula only applies to regular, equiangular, and equilateral figures. The *apothem* is the distance from a side of the polygon to the center of a circle circumscribed around it.

$$A = \frac{1}{2}ap$$

$$A = \frac{1}{2}(6)(75)$$

$$A = 225 \text{ in}^2 \quad \text{Answer}$$

Example:

The area of a rhombus is 96 m^2 . If the length of one diagonal is 12, find the length of the second diagonal.

Two area formulas may be used for a rhombus:

$$A = bh \text{ or } A = \frac{1}{2}d_1d_2$$

Since you are given the length of one diagonal, it makes sense to use the formula that includes diagonals.

$$96 = \frac{1}{2}(12)d_2$$

$$96 = 6d_2$$

$$d_2 = 16 \text{ m} \quad \text{Answer}$$

Area vs. Perimeter

Perimeter and area are NOT related. If two polygons have the same perimeter, they do not necessarily have the same area. Think of two rectangles with a perimeter of 100 cm. The dimensions of the first rectangle could be 30×20 cm, while the dimensions of the second rectangle could be 25×25 cm. Which one has a greater area?

The area of the first rectangle:

$$A = 30(20)$$

$$A = 600 \text{ cm}^2$$

The area of the second rectangle:

$$A = 25(25)$$

$$A = 625 \text{ cm}^2$$

The rectangle measuring 25×25 cm encloses a bigger region. (You may have noticed that this “rectangle” is actually a square. Remember that squares can be thought of as equilateral rectangles.)

Area Ratio of Similar Figures

Remember that similar polygons have the same shape but different size. The ratio of the length of a part of one polygon to the length of the corresponding part of another polygon is called the *scale factor*. Take a look at two equilateral triangles. All equilateral triangles are similar because they have congruent corresponding angles. (Remember, AA proves similarity.) The sides of the first triangle measure 4 cm and the sides of the second triangle measure 8 cm. The scale factor of the two triangles is 4:8, or 1:2. Now let's compare their areas:

$$\text{Area of triangle 1} = \frac{s^2\sqrt{3}}{4}$$

$$A = \frac{4^2\sqrt{3}}{4}$$

$$A = 4\sqrt{3} \text{ cm}^2$$

$$\text{Area of triangle 2} = \frac{s^2\sqrt{3}}{4}$$

$$A = \frac{8^2\sqrt{3}}{4}$$

$$A = 16\sqrt{3}$$

The ratio of the areas of the two triangles is $4\sqrt{3}:16\sqrt{3}$, which simplifies to 1:4. Note that the sides of the triangles have the same ratio as the scale factor, 1:2.

The ratio of the areas is the square of the scale factor, $1^2:2^2$ or 1:4. This is always the case. If the scale factor of two similar figures is $m:n$, then the ratio of their sides and perimeters is also $m:n$ and the ratio of their areas is $m^2:n^2$.

Example:

$ABCDEF \sim KLMNOP$. If the perimeter of the first hexagon is 66 ft and the perimeter of the second is 72 ft, what is the ratio of their areas?

The scale factor of the two hexagons will be the same as the ratio of their perimeters.

66:72 simplifies to 11:12

The ratio of their areas is therefore $11^2:12^2$.

121:144 Answer

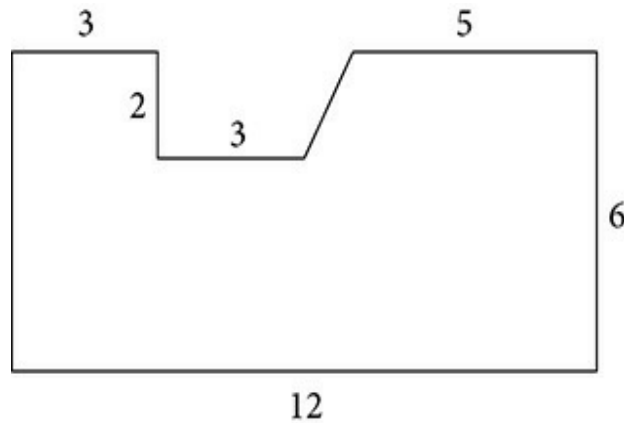
Figures That Combine Numerous Shapes

Of course, the area problems are not as straightforward as some of the examples presented in this chapter. Instead of simply being given a rectangle and asked to calculate its area using $A = bh$ or being given a triangle and asked to calculate its area using $A = \frac{1}{2}bh$, you will need to recognize common figures hidden in diagrams.

The *Area Addition Postulate* states that if a region can be divided into nonoverlapping parts, the area of the given region is the sum of the areas of its parts. This is a useful concept to use when confronted with an irregular shape whose area you may not know how to calculate.

Example:

Find the area of the figure.



The best way to solve this problem is to recognize that the big shape is a rectangle and the missing piece is a right trapezoid. The trapezoid missing base measures $12 - (3 + 5)$, or 4 cm. To solve, find the area of the rectangle and subtract the area of the trapezoid.

$$A = 12(6) - \frac{1}{2}(b_1 + b_2)h$$

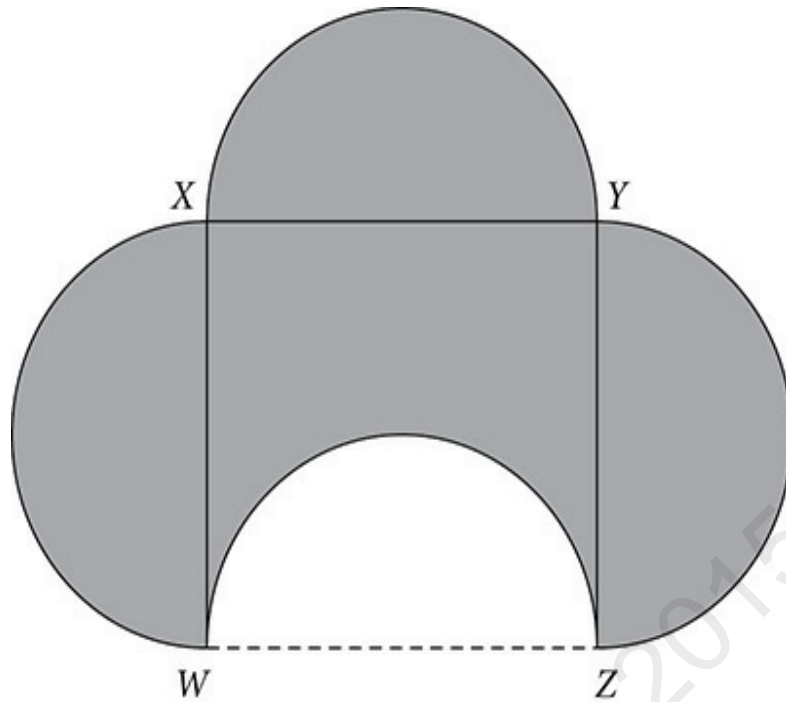
$$A = 72 - \frac{1}{2}(4 + 3)2$$

$$A = 72 - 7$$

$$A = 65 \text{ cm}^2 \quad \text{Answer}$$

Example:

WXYZ is a square of area 36 cm^2 . What is the area of the shaded region?



First, recognize that each side of the square measures 6 cm, since $6^2 = 36$. Before fully attacking the problem, take a look at the given information and come up with a plan. Notice that there are three semicircles included in the shaded region and one semicircle “cut out” of the square. One semicircle’s area will cancel with the semicircle that is *not* included in the shaded region. You need to take the area of the square and add the area of the two remaining semicircles. Two semicircles, of course, form one circle. Simply take the area of the square and add the area of a circle with diameter 6.

$$A = 36 + \pi 3^2$$

$$A = 36 + 9\pi \text{ cm}^2 \quad \text{Answer}$$

Example:

What is the area of a square inscribed in a circle whose diameter is $9\sqrt{2}$ inches?

Recognize that the diameter of the circle is also a diagonal of the square. A square’s diagonal creates two isosceles right triangles within the square, and isosceles right triangles have angles measuring 45° - 45° - 90° .

Do you remember the special relationships among the sides of a $45^\circ\text{-}45^\circ\text{-}90^\circ$ triangle? The diagonal is the hypotenuse of both right triangles, so the legs of the triangles must measure $\frac{9\sqrt{2}}{\sqrt{2}}$, or 9 inches. A square with sides of 9 has an area of 9^2 square inches.

81 square inches Answer

REVIEW QUESTIONS

1. Which of the following is a possible value of the perimeter of the triangle shown in Figure 1?

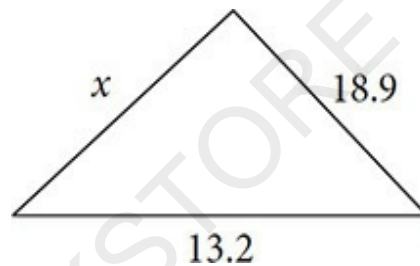


Figure 1

- (A) 35.5
(B) 37.8
(C) 44.4
(D) 64.2
(E) 67.7
2. How many yards does Noah's bicycle travel if the wheels have a radius of 11.75 inches and they rotated 1,010 times?
- (A) 1,035.63 Yards
(B) 2,071.27 Yards
(C) 6,213.81 Yards
(D) 37,282.00 Yards

(E) 74,565.70 Yards

3. A rectangular dining table that measures $38'' \times 78''$ has a semicircular extension at each end, as shown in Figure 2. What is the area of the tabletop to the nearest inch?



Figure 2

- (A) 1,134 Inches
(B) 2,964 Inches
(C) 3,531 Inches
(D) 4,098 Inches
(E) 5,232 Inches
4. What is the area of the shaded region of Figure 3 if \overline{AB} is the diameter of the semicircle and $\overline{AC} = \overline{BC}$?

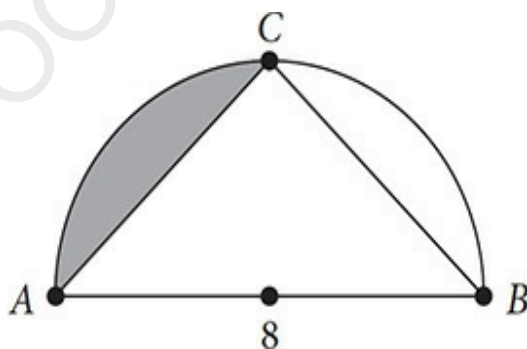


Figure 3

- (A) $\pi - 2$
(B) $2(\pi - 2)$
(C) $4(\pi - 2)$
(D) $8(\pi - 2)$

- (E) $4(2\pi - 1)$
5. In an isosceles right triangle with legs of length x and hypotenuse y , what is the area of the triangle in terms of x ?
- (A) $\frac{1}{2}x^2$
- (B) x^2
- (C) $2x$
- (D) $x^2\sqrt{2}$
- (E) $\frac{\sqrt{2}}{3}x^2$

ANSWERS AND EXPLANATIONS

1. (C) The missing side must be between the sum of the other two sides and the difference between the other two. $5.7 < x < 32.1$. The given measurements for the perimeter are 18.9 and 13.2, so $18.9 + 13.2 + x = \text{perimeter}$. $32.1 + x = \text{perimeter}$. Use the range for x to add in the perimeter. $37.8 < p < 64.2$. The only choice within this range is choice C.
2. (B) One rotation is the circumference of the wheel. $C = \pi d$ and $d = 23.5$ inches, so $\pi d = 23.5\pi$. Multiply 23.5π by 1,010 revolutions. $(23.5)(\pi)(1,010) = 74,565.7016$ inches.
- Divide 74,565.7016 inches by 36 inches to get the answer in yards.
- $74,565.7016 \div 36 = 2,071.27$ yards.
3. (D) To find the area, find the area of the rectangle plus the 2 semicircles (1 complete circle). The rectangle is $38'' \times 78'' = 2,964''$. The diameter of the circle is the same as the side of the rectangle: $38''$. The radius is $19''$, so the area of the circle is $\pi r^2 = \pi (19^2) = 361\pi$. Add 361π to 2,964. 2964

$$+ 361\pi = 2,964 + 1,134.11 = 4,098.11.$$

4. (C) To find the area of the shaded region, find the area of the semicircle, subtract the area of the triangle, then divide by 2. Since the diameter of the semicircle is 8, the radius is 4. The area, $A = \pi r^2 = 16\pi$. Divide the area by 2 for the semicircle = 8π .

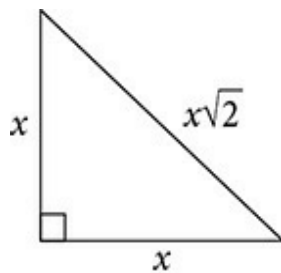
Any triangle inscribed in a semicircle is a right triangle and $\overline{AC} = \overline{BC}$, so the triangle is an isosceles right triangle. If you draw a line from the center of the circle to point C, the line is a radius as well as the height of triangle.

$$A = \frac{1}{2}bh = \frac{1}{2}(8)(4) = 16.$$

The area of the shaded region then is $\frac{8\pi - 16}{2} = \frac{8(\pi - 2)}{2} = 4(\pi - 2)$.

5. (A) For an isosceles right triangle, the sides are in the ratio $x : x : x\sqrt{2}$

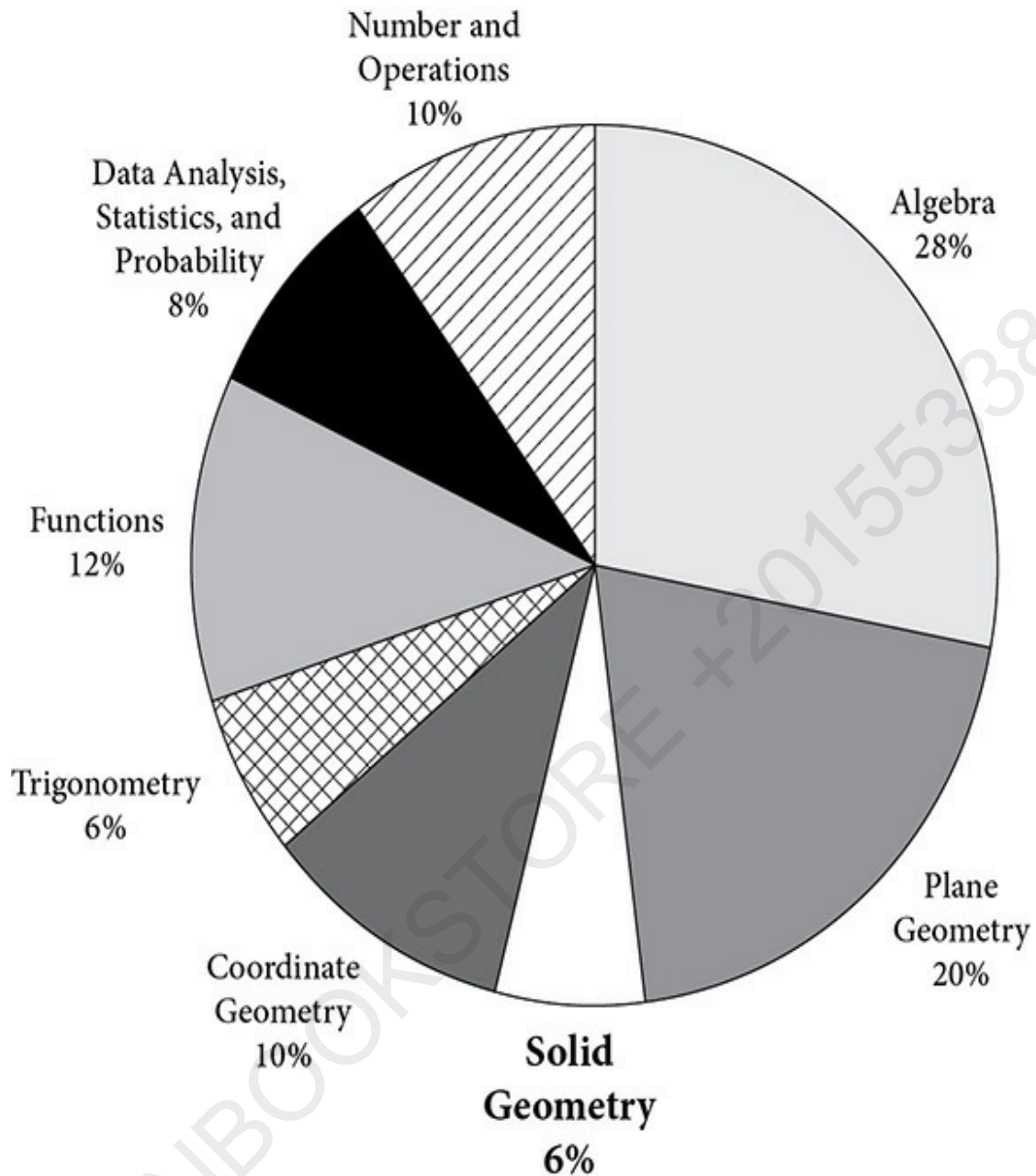
$$A = \frac{1}{2}bh = \frac{1}{2}(x)(x) = \frac{1}{2}x^2$$





Solid Geometry

This chapter provides a review of solid (i.e., three-dimensional) geometry principles. Four to six percent of the Level 1 test questions relate specifically to solid geometry. By definition, *solid geometry* focuses on three-dimensional shapes and figures. The pie chart shows approximately how much of the Level 1 test is related to solid geometry:



The following solid geometry topics are covered in this chapter:

1. Vocabulary for Polyhedra
2. Prisms
 - a. Surface Area
 - b. Lateral Surface Area
 - c. Volume
 - d. Distance Between Opposite Vertices of a Rectangular Prism
3. Cylinders

- a. Surface Area
 - b. Lateral Surface Area
 - c. Volume
4. Pyramids
 - a. Surface Area
 - b. Lateral Surface Area
 - c. Volume
 5. Cones
 - a. Surface Area
 - b. Lateral Surface Area
 - c. Volume
 6. Spheres
 - a. Surface Area
 - b. Volume
 7. Volume Ratio of Similar Figures

The following reference information is provided during the Level 1 test. You do NOT need to memorize these formulas. Become familiar with the information below so that you can refer to it during the test, if necessary. These formulas are printed in the directions of the test.

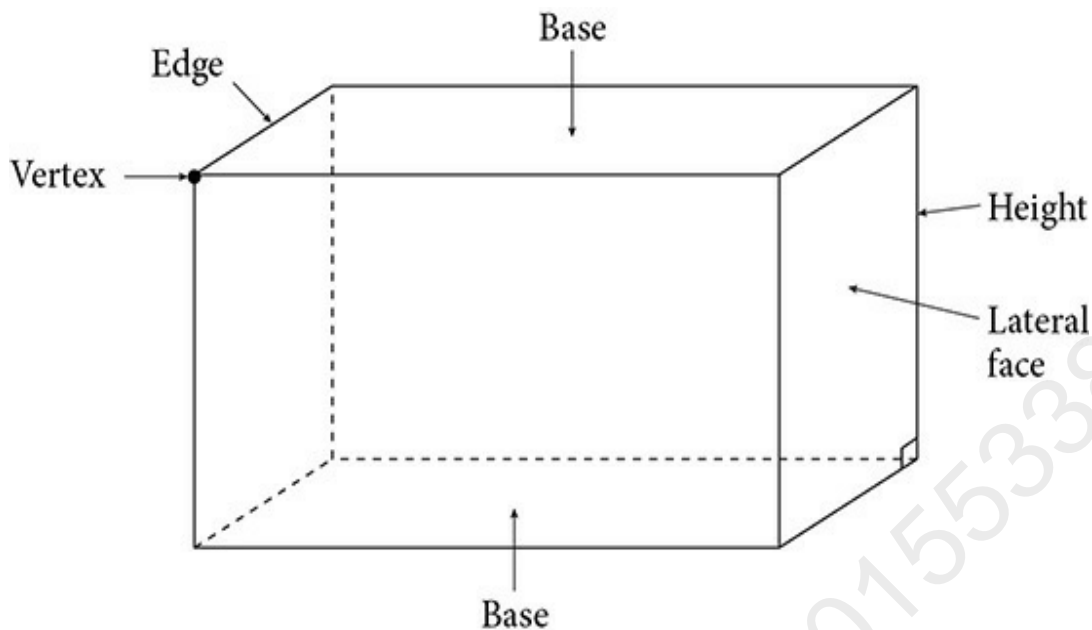
REFERENCE INFORMATION	
Right circular cone with radius r and height h	Volume = $\frac{1}{3}\pi r^2 h$
Right circular cone with circumference of base C and slant height ℓ	Lateral Area = $\frac{1}{2}C\ell$
Sphere with radius r	Volume = $\frac{4}{3}\pi r^3$ Surface Area = $4\pi r^2$
Pyramid with base area B and height h	Volume = $\frac{1}{3}Bh$

Vocabulary for Polyhedra

The previous chapter focused on plane, or two-dimensional, figures. Remember the term “polygon” from plane geometry—a many-sided closed figure created by connecting line segments endpoint to endpoint. A *polyhedron* is a many-sided solid created by connecting polygons along their sides. A polyhedron encloses a single region of space, and the plural of polyhedron is polyhedra.

The parts of a polyhedron are as follows:

- **Faces.** The flat surfaces of a polyhedron that are shaped like polygons
- **Edge.** A segment where two faces intersect
- **Vertex.** The point of intersection of three or more edges
- **Base** (of a prism or cylinder). The two congruent, parallel faces
- **Base** (of a pyramid or cone). The circular (for a cone) or polygonal (for a pyramid) face that does not contain the common vertex
- **Lateral faces.** The face(s) that make up the sides of the solid (for prisms, the lateral faces are always parallelograms).
- **Altitude.** The segment perpendicular to the plane of both bases (for a prism or cylinder); the perpendicular segment joining the vertex to the plane of the base (for a pyramid or cone)
- **Height** (h). The length of the altitude
- **Slant height** (ℓ). The distance from the edge of the base to the common vertex



Some solids, such as prisms and pyramids, have flat faces, while others, such as cones, cylinders, and spheres, have curved faces. Solids with curved faces are not polyhedra, however, since they are not created by connecting polygons along their sides.

Let's review a few terms that are commonly used in solid geometry problems:

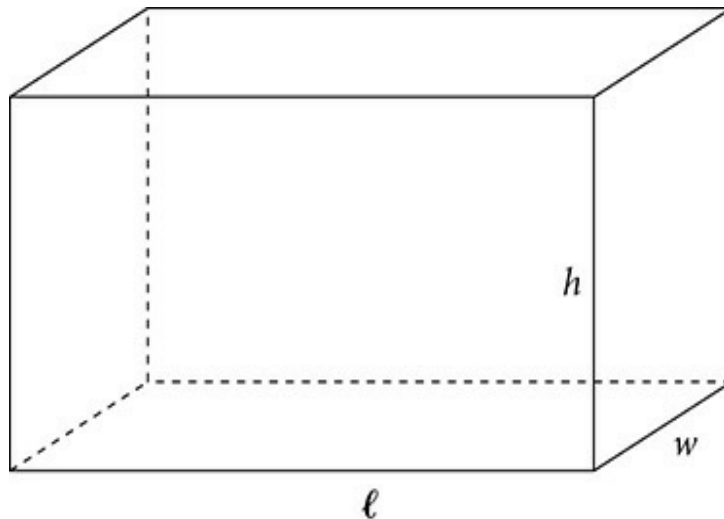
Volume. The amount of space enclosed by a solid.

Surface area. The sum of the area of all of the faces.

Lateral surface area. The sum of the area of only the lateral faces (or the sides). This is also referred to as simply the lateral area.

Prisms

A *prism* is a polyhedron consisting of two congruent, parallel bases connected by lateral faces shaped like parallelograms. Prisms are classified by their bases: *Rectangular prisms* have bases shaped like rectangles, *triangular prisms* have bases shaped like triangles, *hexagonal prisms* have bases shaped like hexagons, and so on. The rectangular prism is the most commonly used solid on the Level 1 test. A rectangular prism looks like what you would think of as a box.



Rectangular Prism

SURFACE AREA	LATERAL SURFACE AREA	VOLUME
$S = 2\ell w + 2\ell h + 2hw$	$S = 2\ell h + 2hw$	$V = \ell wh$

In general, the formula for the volume for any prism can be written as:

$$V = Bh$$

where B = the area of the base and h = the height.

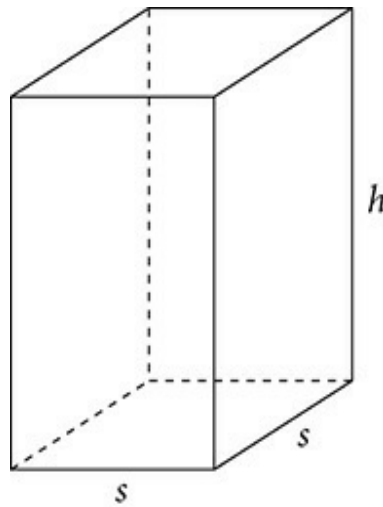
In the case of a rectangular prism, the area of the base is the product of its length and width, ℓw . Substituting ℓw for B results in $V = \ell wh$. The volume of a cube with edge s is therefore $V = Bh = (s^2)s = s^3$. Volume is measured in *cubic units*, such as centimeters³, inches³, feet³, and meters³.

Example:

The volume of a square prism is equal to its lateral surface area. Find the length of the sides of the base.

Note that a square prism is not equivalent to a cube. The two bases of this prism must be squares, but the height may be any length. A square prism is equivalent to a rectangular prism whose length and width have equal

measures. Draw a diagram to help solve this problem:



Let s equal the sides of the base and h equal the height of the prism. Use formulas for a rectangular prism and substitute s for the length and width. The volume is

$$V = \ell wh = s^2 h$$

The lateral surface area is therefore

$$S = 2\ell h + 2hw = S = 2sh + 2hs = 4sh$$

Now set the volume equal to the lateral surface area and solve for s :

$$s^2 h = 4sh \quad \text{Divide both sides by } h. \text{ This is okay to do since } h \text{ is a length and cannot equal zero.}$$

$$s^2 = 4s \quad \text{Divide both sides by } s, \text{ since } s \text{ also cannot equal zero.}$$

$$s = 4 \quad \text{Answer}$$

Example:

The surface area of a cube is 150 cm^2 . Find the length of its edges.

The formula for the surface area of a cube is similar to the formula for a rectangular prism, except length, width, and height are equal in a cube. If the length of an edge of the cube is s , $s = \ell = w = h$.

$$S = 2\ell w + 2\ell h + 2hw \text{ becomes } S = 2s^2 + 2s^2 + 2s^2 = 6s^2.$$

$$150 = 6s^2$$

$$25 = s^2$$

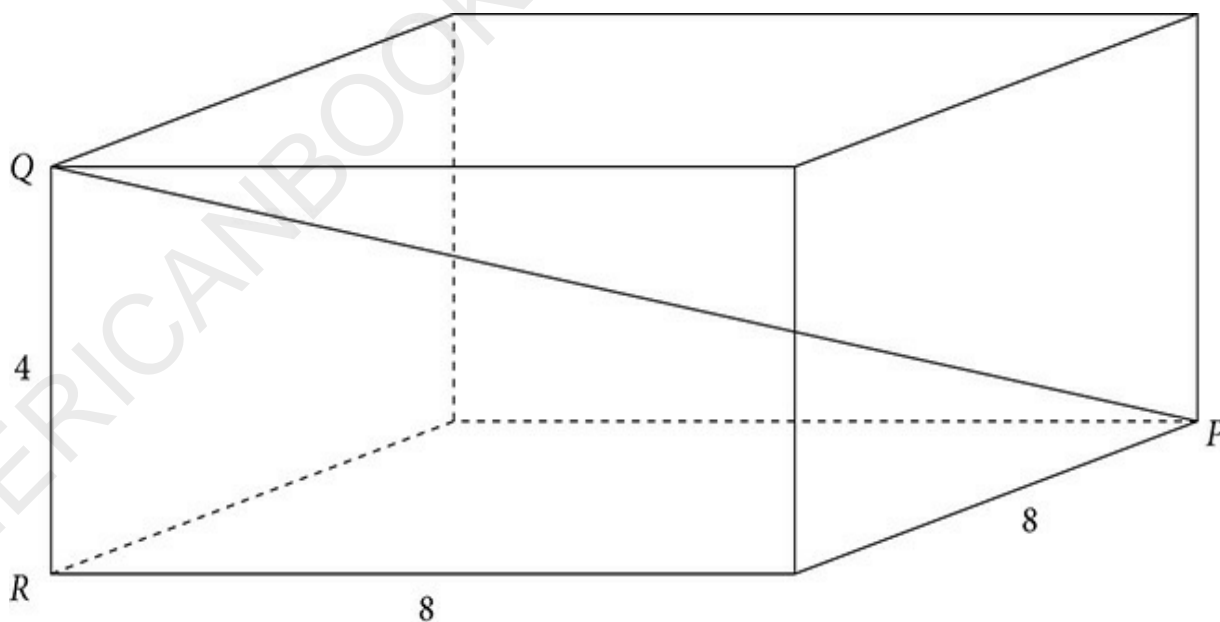
$$s = 5 \text{ cm} \quad \text{Answer}$$

Distance Between Opposite Vertices of a Rectangular Prism

Along with surface area and volume, you may be asked to determine the distance between opposite vertices of a rectangular prism. The problem is included here in the Solid Geometry chapter because it involves a prism, but it can actually be solved by using the Pythagorean Theorem *twice*.

Example:

Find the distance from vertex P to vertex Q in the rectangular prism below.



Think of \overline{PQ} as the hypotenuse of a right triangle whose legs are the

diagonal of the base and the edge of the prism, \overline{QR} . First find the length of the diagonal of the base:

$$\overline{RP}^2 = 8^2 + 8^2$$

$$\overline{RP}^2 = 128$$

$$\overline{RP} = \sqrt{128} = 8\sqrt{2}$$

Did you recognize that \overline{RP} is the hypotenuse of an isosceles right triangle? If so, you can determine the length, \overline{RP} simply by remembering the ratio of the sides of a 45° - 45° - 90° triangle.

Now, use the Pythagorean Theorem for a second time to determine PQ .

$$\overline{PQ}^2 = (8\sqrt{2})^2 + 4^2$$

$$\overline{PQ}^2 = 64(2) + 16$$

$$\overline{PQ}^2 = 128 + 16 = 144$$

$$\overline{PQ} = \sqrt{144}$$

$$\overline{PQ} = 12 \quad \text{Answer}$$

Another way to find the distance between opposite vertices of a rectangular prism is to use the formula:

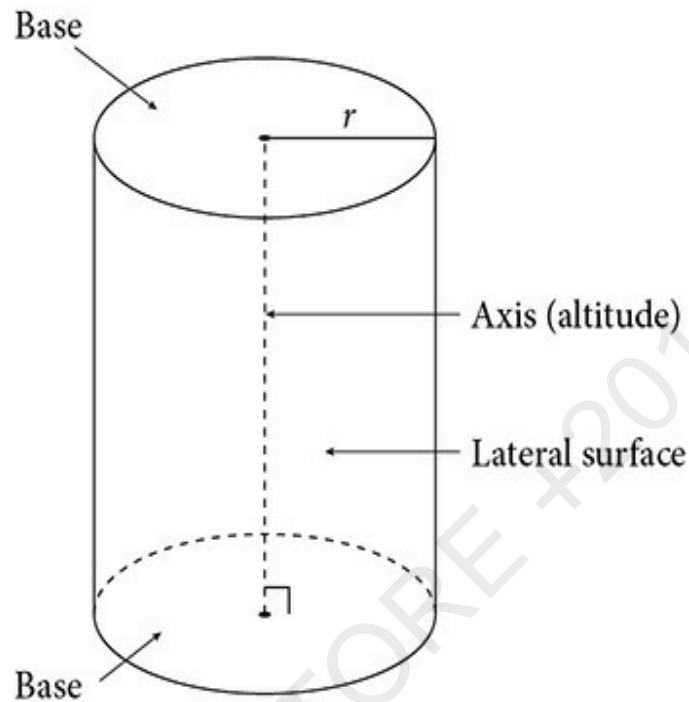
$$\text{Distance} = \sqrt{\ell^2 + w^2 + h^2}$$

Try using this formula for the last example to get:

$$\text{Distance} = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{144} = 12$$

Cylinders

A *cylinder* is similar to a prism with circular bases. Right circular cylinders are the most commonly used cylinders on the Level 1 test. They consist of two congruent, parallel, circular bases joined by an *axis* that is perpendicular to each. The axis of a right circular cylinder is also its *altitude*.



Cylinder

SURFACE AREA	LATERAL SURFACE AREA	VOLUME
$S = 2\pi r^2 + 2\pi rh$	$S = 2\pi rh$	$V = \pi r^2 h$

In general, the formula for the volume for any cylinder is the same as that for any prism and can be written as:

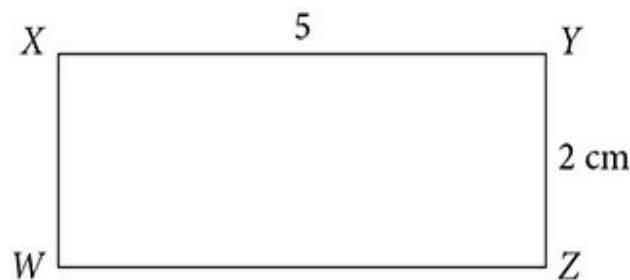
$$V = Bh$$

where B = the area of the base and h = the height.

In the case of a right circular cylinder, the area of the base is πr^2 . Substituting πr^2 for B results in $V = \pi r^2 h$.

Example:

Find the volume of the solid created by rotating rectangle $WXYZ$ 360° around side \overline{XY} .



The hardest part of a problem like this is to visualize the solid formed by rotating the rectangle. Picture rectangle $WXYZ$ rotating fully around the axis XY . A right circular cylinder is created whose height is 5 cm (the length XY) and whose radius is 2 cm. (You may think of this cylinder as being “on its side,” since the axis is not vertical.)

Using $r = 2$ cm and $h = 5$ cm, its volume is

$$V = \pi(2)^2(5)$$

$$V = 20\pi \text{ cm}^3 \quad \text{Answer}$$

Example:

Find the total surface area of a cylinder whose volume equals its lateral surface area and whose height is 10 inches.

First, set the formula for lateral surface area equal to that of volume to solve for r . Since lateral surface area equals $2\pi rh$ and volume equals $\pi r^2 h$, the resulting equation is

$$2\pi rh = \pi r^2 h$$

Notice that h cancels, meaning that height is not a factor in a cylinder having equal volume and lateral surface area. One r and π also cancel to get

$$2 = r$$

Use the given information, $h = 10$, and the radius that you just found, $r =$

2, to get the cylinder's total surface area.

$$S = 2\pi r^2 + 2\pi rh$$

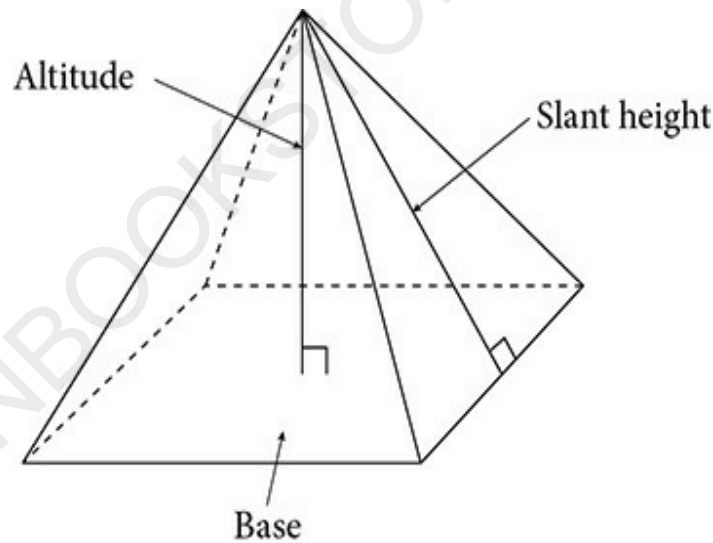
$$S = 2\pi(2)^2 + 2\pi(2)(10)$$

$$S = 8\pi + 40\pi$$

$$S = 48\pi \text{ in}^2 \quad \text{Answer}$$

Pyramids

A *pyramid* consists of one base and triangular lateral faces that connect at a common vertex. Like prisms, pyramids are classified by their base: Rectangular pyramids have a base shaped like a rectangle, triangular pyramids have a base shaped like a triangle, hexagonal pyramids have a base shaped like a hexagon, and so on.



The following are formulas for a *regular pyramid*. A pyramid is regular if its base is a regular polygon (i.e., a square or equilateral triangle) and its lateral edges are congruent.

Regular Pyramid

SURFACE AREA	LATERAL SURFACE AREA	VOLUME
$S = \frac{1}{2}\ell P + B$	$S = \frac{1}{2}\ell P$	$V = \frac{1}{3}Bh$
$B =$ area of the base	$\ell =$ slant height $P =$ perimeter of the base	

In general the formula for the volume for any pyramid can be written as:

$$V = \frac{1}{3}Bh$$

where $B =$ the area of the base and $h =$ the height.

In the case of a rectangular pyramid, the area of the base is the product of its length and width, ℓw . Substituting ℓw for B results in $V = \frac{1}{3} \ell w h$.

Remember that the volume formula for a pyramid, $V = \frac{1}{3}Bh$, is listed in the reference information of the Level 1 test.

Example:

Find the volume of a pyramid whose base is an equilateral triangle with sides of length 4 cm and whose height is 9 cm.

Start by finding B , the area of the base. Recall that the area of an equilateral triangle is

$$A = \frac{s^2\sqrt{3}}{4}$$

$$A = \frac{4^2\sqrt{3}}{4}$$

$$A = 4\sqrt{3} \text{ cm}^2$$

Since the volume of a pyramid is given by the formula $V = \frac{1}{3}Bh$ and $h = 9$ cm, the volume is

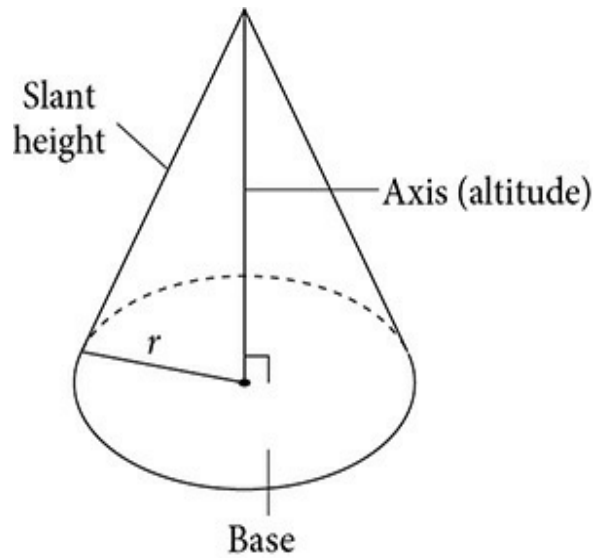
$$V = \frac{1}{3}Bh$$

$$V = \frac{1}{3}(4\sqrt{3})(9)$$

$$V = 12\sqrt{3} \text{ cm}^3 \quad \text{Answer}$$

Cones

A *cone* consists of one circular base and a lateral surface that comes to a common vertex. Right circular cones are the most commonly used cones on the Level 1 test. They consist of a circular base connected to a vertex by an *axis* perpendicular to the base. The axis of a right circular cone is also its *altitude*.



Right Circular Cone

SURFACE AREA	LATERAL SURFACE AREA	VOLUME
$S = \frac{1}{2}c\ell + \pi r^2$	$S = \frac{1}{2}c\ell$	$V = \frac{1}{3}\pi r^2 h$

ℓ = slant height
 c = circumference of the base

In general, the formula for the volume for any cone is the same as that for any pyramid and can be written as:

$$V = \frac{1}{3}Bh$$

where B = the area of the base and h = the height.

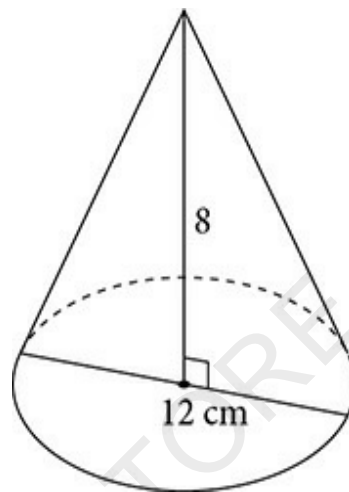
In the case of a right circular cone, the area of the base is πr^2 . Substituting πr^2 for B results in $V = \frac{1}{3}\pi r^2 h$.

Remember that the volume formula for a right circular cone, $V = \frac{1}{3}\pi r^2 h$,

and the lateral surface area formula, $S = \frac{1}{2}c\ell$, are listed in the reference information of the Level 1 test. You don't need to memorize these!

Example:

Find the lateral area of a cone whose diameter is 12 cm and whose height is 8 cm.



You need to solve for the circumference, C , and the slant height, ℓ , in order to use the formula for lateral surface area.

Circumference can be determined using $C = \pi d$ or $C = 2\pi r$. Since you're given the diameter, it makes sense to use the first formula:

$$C = \pi(12)$$

The radius is therefore half the diameter, or 6 cm. Notice that the radius and the altitude form a right triangle whose hypotenuse is the slant height. Use the Pythagorean Theorem to solve for ℓ .

$$\ell^2 = 8^2 + 6^2$$

$$\ell^2 = 100$$

$$\ell = 10$$

Now you have enough information to solve for the lateral area.

$$S = \frac{1}{2}C\ell$$

$$S = \frac{1}{2}(12\pi)(10)$$

$$S = 60\pi \text{ cm}^2 \quad \text{Answer}$$

Example:

A right circular cylinder and a right circular cone have the same radius and volume. If the cone has a height of 18 inches, find the height of the cylinder.

To solve for the height, set the volume of each solid equal to each other:

$$\pi r^2 h = \frac{1}{3} \pi r^2 (18)$$

$$h = 6 \text{ inches} \quad \text{Answer}$$

The height of the cylinder must be one-third the height of the cone, since the volume of the cone is one-third the volume of the cylinder.

Example:

A right circular cone can have a cross section in the shape of all of the following EXCEPT:

- (A) a circle
- (B) a triangle
- (C) a rectangle
- (D) its base
- (E) an ellipse

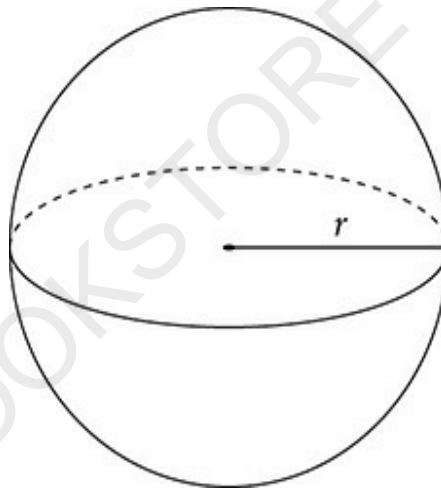
A *cross section* of a solid is a figure formed when a plane intersects the solid. Try to visualize what happens when you cut a cone. A circle is

obviously possible when the plane intersecting the cone is parallel to the base. The base is a possibility if the plane intersecting the cone is the same plane that contains the base. A triangle is possible if the plane cuts through the vertex of the cone and is perpendicular to the base. An ellipse results if the plane that intersects the cone is at an angle, creating an oval-type figure. It is not possible to have a cross-sectional area in the shape of a rectangle.

C is the correct answer. Answer

Spheres

A *sphere* is the set of all points in space at a given distance from a given point. Rotating a circle 360° around one of its diameters creates a sphere.



Sphere

SURFACE AREA	VOLUME
$S = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

Remember that the volume formula for a sphere, $V = \frac{4}{3}\pi r^3$, and the

surface area formula, $S = 4\pi r^2$, are listed in the reference information of the Level 1 test. Again, you don't need to memorize these!

Example:

Find the volume of a sphere whose surface area equals its volume.

First, set surface area equal to volume and solve for r .

$$4\pi r^2 = \frac{4}{3}\pi r^3$$

$$4 = \frac{4}{3}r$$

$$r = 3$$

Now use the radius to solve for volume:

$$V = \frac{4}{3}\pi(3)^3$$

$$V = 36\pi \quad \text{Answer}$$

Example:

A ball is immersed in a cup of water and displaces 288π cubic units of water. Find the surface area of the ball.

The amount of water displaced by the ball is equivalent to its volume. Therefore, you have enough information to write an equation for the volume of the ball and solve for r .

$$\frac{4}{3}\pi r^3 = 288\pi$$

$$r^3 = \frac{3}{4}(288)$$

$$r^3 = 216$$

$$r^3 = \sqrt[3]{216}$$

$$r = 6$$

Now substitute $r = 6$ into the surface area formula to get

$$S = 4\pi(6)^2$$

$$S = 144\pi \text{ units}^2 \quad \text{Answer}$$

Volume Ratio of Similar Figures

Solids are said to be similar if their bases are similar and their corresponding parts are proportional. Just like similar polygons mentioned in the Plane Geometry chapter, *similar solids* have the same shape but different size. A square, right pyramid with a base area of 2 cm^2 , for example, is similar to a square, right pyramid with a base area of 4 cm^2 . The ratio of the length of corresponding parts of similar solids is called the *scale factor*.

If the scale factor of two similar solids is $m:n$, then the following ratios are true:

1. The ratio of their corresponding base perimeters or circumferences is also $m:n$.
2. The ratio of their base areas, total surface areas, and lateral surface areas is $m^2:n^2$.
3. The ratio of their volumes is $m^3:n^3$.

Example:

The lateral surface areas of two similar right cylinders are $80\pi \text{ in}^3$ and $120\pi \text{ in}^3$, respectively. Find the ratio of their volumes.

First determine the scale factor of the similar cylinders by comparing their lateral surface areas.

$80\pi:120\pi$ simplifies to 8:12, which simplifies to 2:3.

Therefore, the scale factor is $\sqrt{2}:\sqrt{3}$.

Now, find the volume ratio by cubing the scale factor:

$$(\sqrt{2})^3:(\sqrt{3})^3$$

$$2\sqrt{2}:3\sqrt{3} \quad \text{Answer}$$

REVIEW QUESTIONS

1. What is the area of a piece of paper needed to form an open cone with a diameter of 18 inches and a slant height of 19.5 inches?
(A) 84.8 in^2
(B) 254.5 in^2
(C) 551.1 in^2
(D) 805.6 in^2
(E) $1,653.6 \text{ in}^2$
2. What is the volume of a cube with surface area of 61.44 cm^2 ?
(A) 10.24 cm^3
(B) 32.77 cm^3
(C) 94.21 cm^3
(D) 126.98 cm^3

- (E) $1,073.74 \text{ cm}^3$
3. How many cubic inches of water can a spherical water balloon hold if its diameter is 12.6 inches?
- (A) 39.7 in^3
(B) 250.0 in^3
(C) 333.4 in^3
(D) 785.5 in^3
(E) $1,047.4 \text{ in}^3$
4. In [Figure 1](#), a 6-foot diameter wire hoop is suspended 8 feet off the ground and a curtain is suspended from the hoop to the ground. What is the area of the curtain?



Figure 1

- (A) 18.8 ft
(B) 25.1 ft
(C) 48.0 ft
(D) 150.8 ft
(E) 201.0 ft
5. A triangular prism is shown in [Figure 2](#). If the length is increased by 30

percent, by how much is the volume increased?

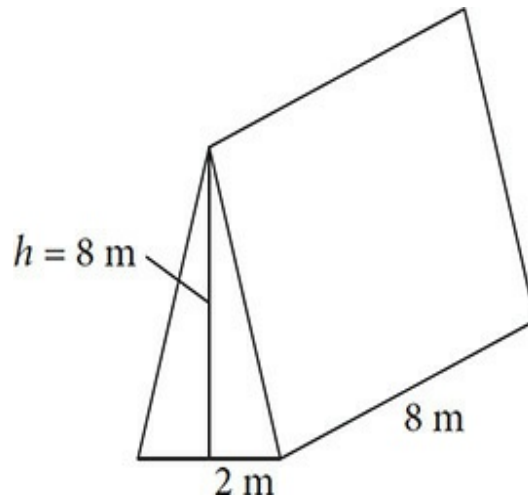


Figure 2

- (A) 19.2 m^3
- (B) 20.8 m^3
- (C) 30.0 m^3
- (D) 64.0 m^3
- (E) 83.2 m^3

ANSWERS AND EXPLANATIONS

1. (C) The surface area of a cone without the base is $SA = \pi rl$, where l is the slant height:

$$SA = \pi (9) (19.5) = 551.1$$

2. (B) A cube has 6 sides, so divide the total surface area by 6 to find the area of one side:

$61.44 \div 6 = 10.24$. Take the square root to find the length of one side:

$$\sqrt{10.24} = 3.2. \text{ Volume of a cube} = s^3 = (3.2)^3 = 32.77 \text{ cm}^3$$

3. (E) The volume of a sphere is $\frac{4}{3}\pi r^3$. If the diameter is 12.6 inches, then

the radius is 6.3 inches:

$$V = \frac{4}{3}\pi(6.3)^3 = \frac{4}{3}\pi(250.047) = 333.396\pi = 1047.4 \text{ in}^3$$

4. (D) Picture the curtain off of the hoop. It is a rectangle 8 feet tall and the length is the circumference of the hoop. $C = \pi d = 6\pi = 18.85$. $A = l w = 18.85 \times 8 = 150.8 \text{ ft}^2$

5. (A) Find the current volume $V = \text{area of triangular base} \times \text{height}$.

$$\text{The area of the base is } \frac{1}{2}bh = \frac{1}{2} \times 2 \times 8 = 8\text{m}^2$$

$$V = 8\text{m}^2 \times 8\text{m} = 64\text{m}^3$$

$$\text{Find the new length: } 130\% \times 8 = 10.4\text{m}$$

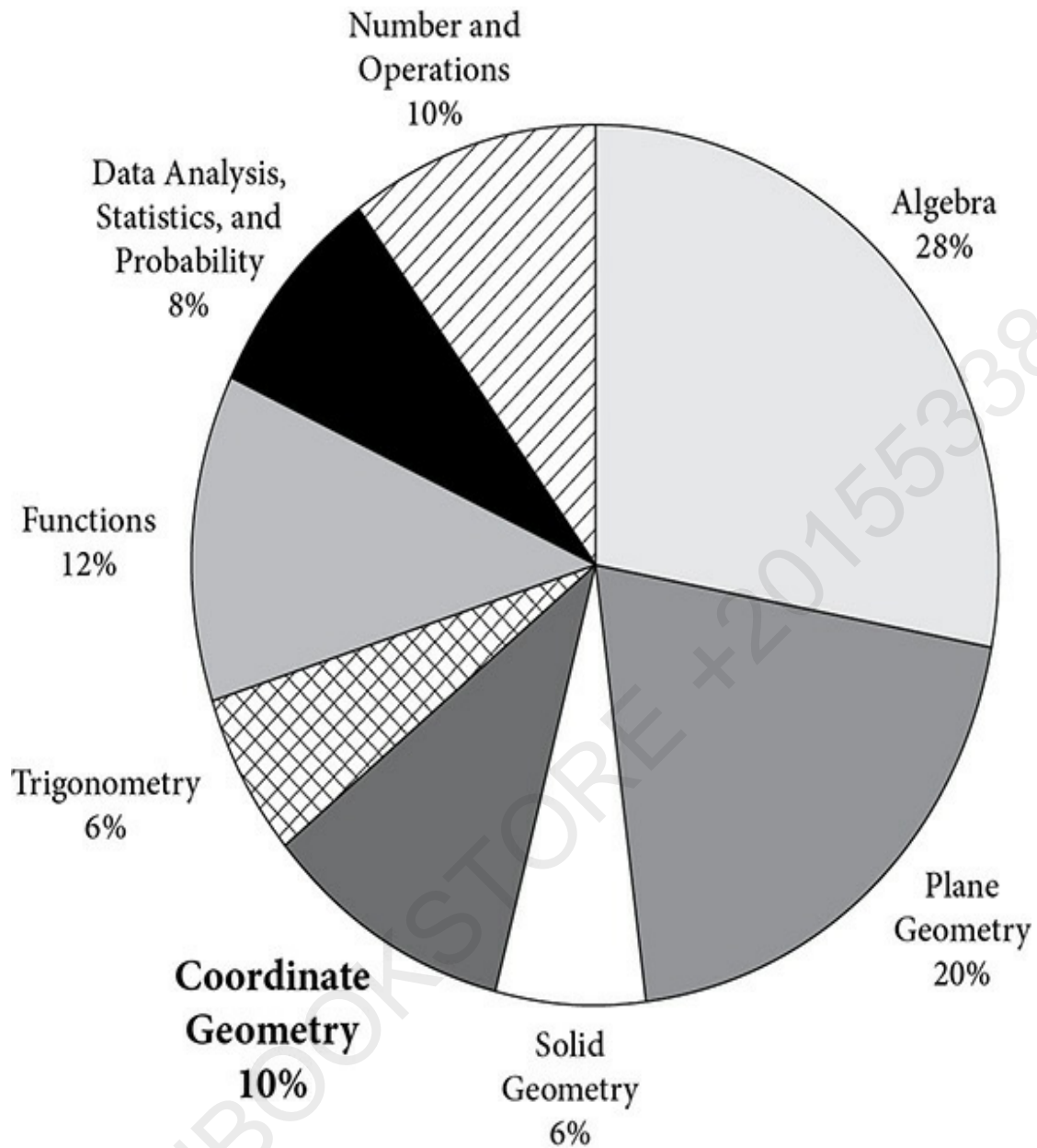
$$\text{Find the new volume: } 8\text{m}^2 \times 10.4\text{m} = 83.2\text{m}^3$$

$$\text{Subtract to find the increase: } 83.2 - 64 = 19.2\text{m}^3$$



Coordinate Geometry

This chapter provides a review of coordinate geometry principles. Eight to twelve percent of the Level 1 test questions relate specifically to coordinate geometry. In actuality, however, about 20 percent of the Level 1 test questions assume an understanding of coordinate geometry concepts. Coordinate geometry focuses on graphing in an xy -coordinate plane. The pie chart shows approximately how much of the Level 1 test is related to coordinate geometry:



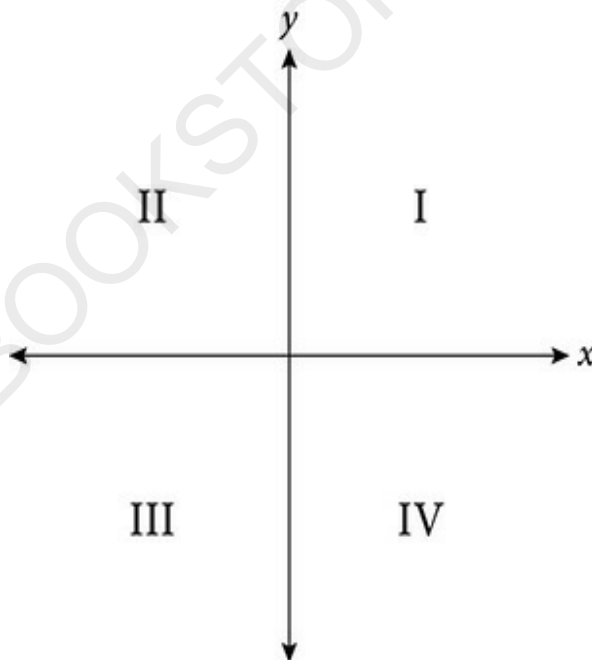
The following geometry topics are covered in this chapter:

1. Plotting Points
2. Midpoint
3. Distance
4. Slope
5. Slope of Parallel and Perpendicular Lines
6. Equations of Lines
 - a. Horizontal and Vertical Lines
 - b. Standard Form

- c. Point-Slope Form
- d. Slope-Intercept Form
- e. Determining x - and y -Intercepts
- 7. Circles
- 8. Parabolas
- 9. Graphing Inequalities
- 10. Graphing Absolute Value

Plotting Points

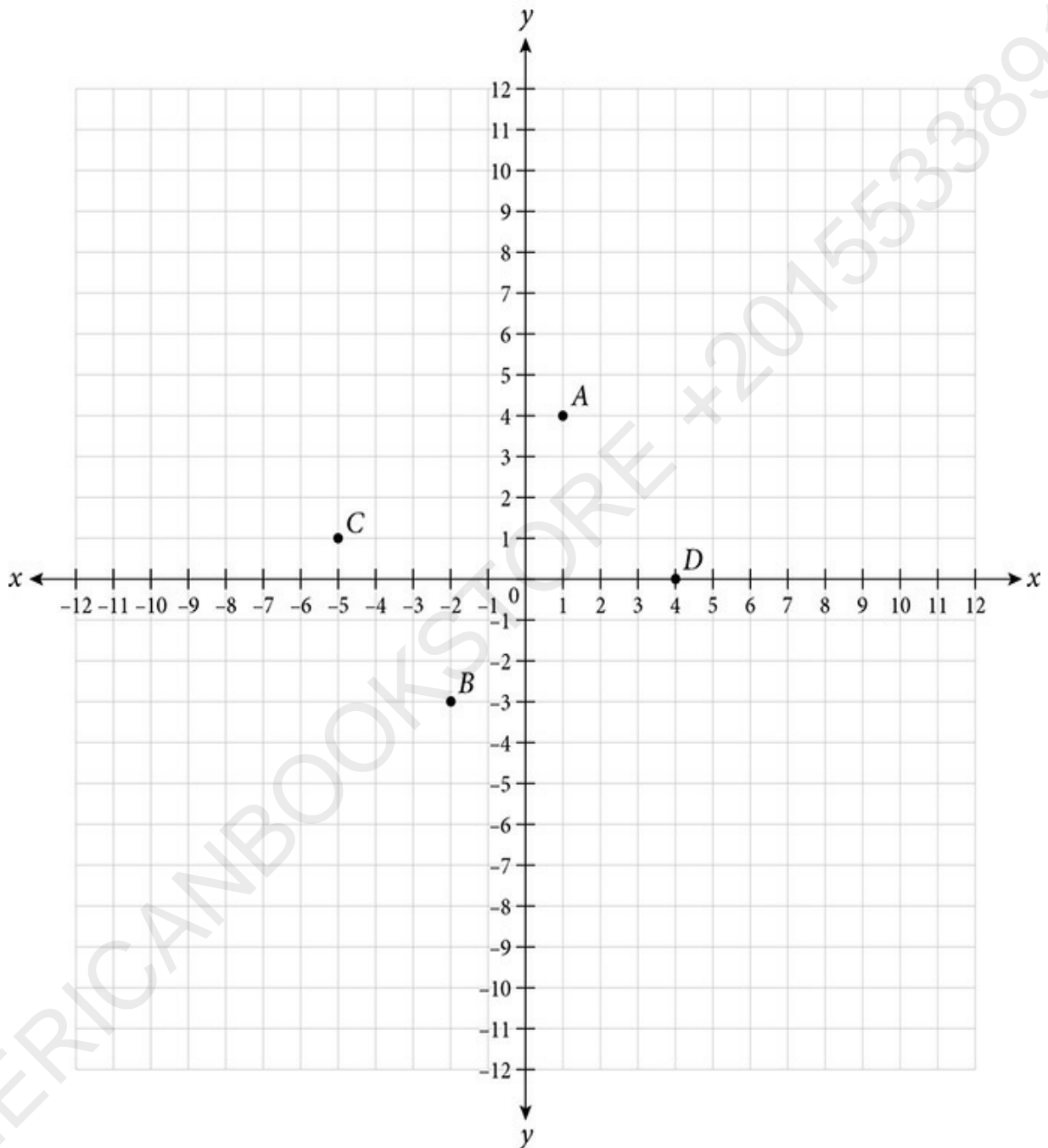
A *plane rectangular coordinate system* is created by drawing two axes that intersect at right angles at an origin. The horizontal axis is called the y -axis, and the vertical axis is called the x -axis. The axes create four *quadrants* in the plane, which are numbered using Roman numerals as shown:



An *ordered pair* (x, y) denotes a unique point in the plane. To plot a point, simply graph the location of its x - and y -coordinates. The x -coordinate of a point is also referred to as the *abscissa*, while the y -coordinate is also referred to as the *ordinate*.

Example:

Graph the following points in the same coordinate plane: $A(1, 4)$, $B(-2, -3)$, $C(-5, 1)$, $D(4, 0)$.



Example:

Find three solutions of $2x + 3y = 1$.

The *graph of an equation in two variables* is the set of all points whose x - and y -coordinates satisfy the equation. To find three solutions to this equation, simply choose three values for x and solve for the corresponding value of y . Write y in terms of x , and create an xy -table to determine solutions of the equation.

x	$y = \frac{1-2x}{3}$	y
0	$y = \frac{1-2(0)}{3}$	$\frac{1}{3}$
2	$y = \frac{1-2(2)}{3}$	-1
-1	$y = \frac{1-2(-1)}{3}$	1

The graph of $2x + 3y = 1$ is a *line*, so there are infinitely many points that will satisfy the equation. Here three possible points are given.

$(0, \frac{1}{3})$, $(2, -1)$, and $(-1, 1)$ are three possible solutions. Answer

Midpoint

The *midpoint* of a segment with endpoints (x_1, y_1) and (x_2, y_2) is the point:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Finding the midpoint of a line segment can also be thought of as finding

the *average* of the x - and y -coordinates.

Example:

Find the midpoint \overline{AB} given the endpoints $A(1, 6)$ and $B(-3, -7)$.

Substitute the given coordinates into the midpoint formula to get

$$\left(\frac{1 + -3}{2}, \frac{6 + -7}{2} \right)$$

$$\left(\frac{-2}{2}, \frac{-1}{2} \right)$$

$$\left(1, \frac{-1}{2} \right) \quad \text{Answer}$$

Example:

M is the midpoint of \overline{AB} . Find the coordinates of B if A has coordinates $(3, 8)$ and M has coordinates $(-4, 0)$.

Let the coordinates of B be (x_2, y_2) ; then:

$$\frac{3 + x_2}{2} = -4 \quad \text{and} \quad \frac{8 + y_2}{2} = 0$$

$$3 + x_2 = -8 \quad 8 + y_2 = 0$$

$$x_2 = -11 \quad y_2 = -8$$

$$B(-11, -8) \quad \text{Answer}$$

Distance

The *distance* between any two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Some aspects of coordinate geometry simply involve solving a plane geometry question by using an xy -coordinate plane. Here's an example of one.

Example:

Find the perimeter of $\triangle ABC$ given its vertices are $A(2, 2)$, $B(-1, 5)$, and $C(-5, 2)$.

This problem requires applying the distance formula three times to find the lengths of the three sides of the triangle. Solve for AB , BC , and AC as follows:

$$\overline{AB} = \sqrt{(2 - (-1))^2 + (2 - 5)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$$

$$\overline{BC} = \sqrt{(-1 - (-5))^2 + (5 - 2)^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\overline{AC} = \sqrt{(2 - (-5))^2 + (2 - 2)^2} = \sqrt{(7)^2 + (0)^2} = \sqrt{49} = 7$$

Notice that \overline{AC} is a horizontal line segment, so the distance from A to C can be easily found by determining the change in the x -coordinates. $|2 - (-5)| = 7$. The perimeter of the triangle is

$$7 + 5 + 3\sqrt{2}$$

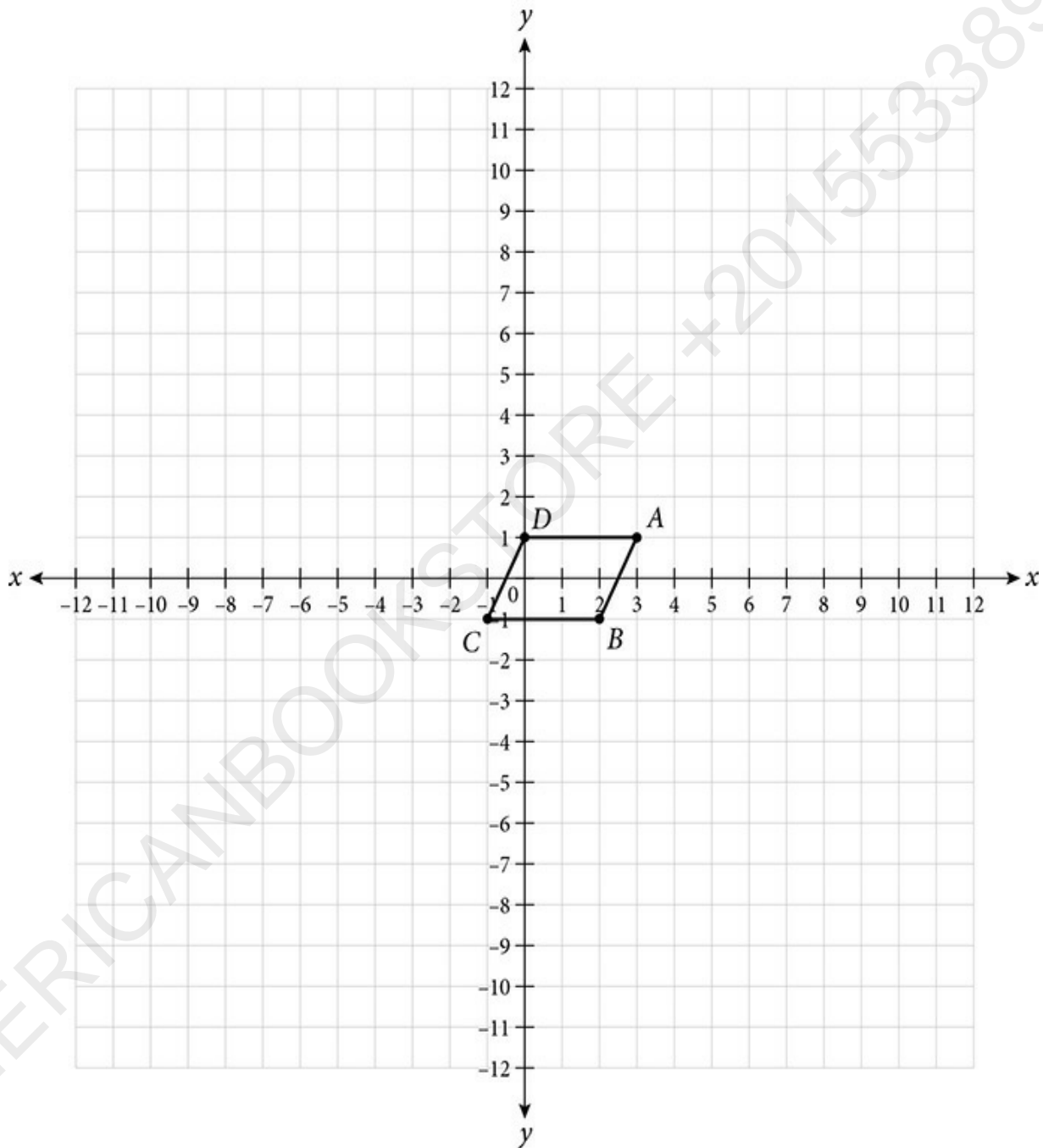
$$12 + 3\sqrt{2} \quad \text{Answer}$$

The concept of distance can also be applied to finding the area of figures in a coordinate plane. Use the distance formula to find the length of the base and/or height of a figure and substitute these values into the appropriate area formula. (See [Chapter 5](#), "Plane Geometry," for a review of area formulas.)

Example:

Find the area of parallelogram $ABCD$ given its vertices are $A(3, 1)$, $B(2, -1)$, $C(-1, -1)$, and $D(0, 1)$.

Sketch a diagram to help picture the parallelogram.



Assume \overline{BC} is the base of the parallelogram and the height is the length of the perpendicular from vertex D to side \overline{BC} . Notice that there is no

change in the y -coordinate in the segment \overline{BC} , so the distance from B to C can be found simply by using absolute value.

$$\overline{BC} = |2 - -1| = 3$$

The height is $|1 - -1| = 2$

The area of the parallelogram is therefore

$$A = bh = 3(2) = 6$$

6 units² Answer

Slope

Slope is the measure of the steepness of a line. The slope of a line containing the points (x_1, y_1) and (x_2, y_2) is

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

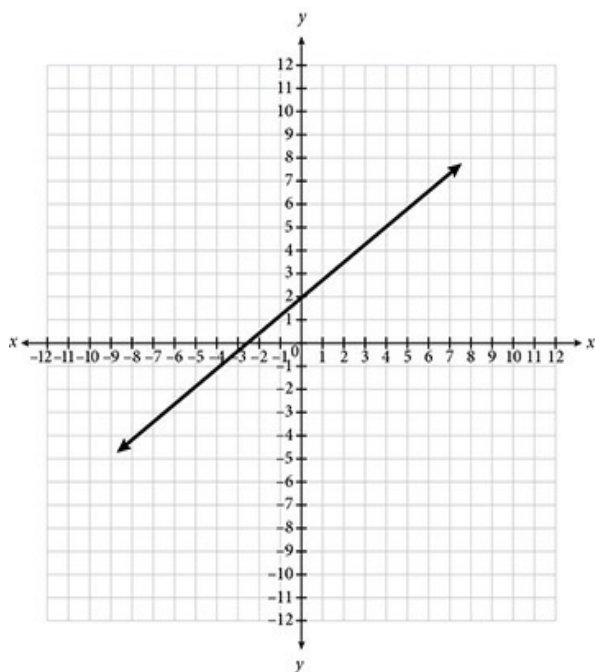
Horizontal lines have no change in y , so the slope of a horizontal line is zero:

$$m = \frac{0}{\text{change in } x} = 0$$

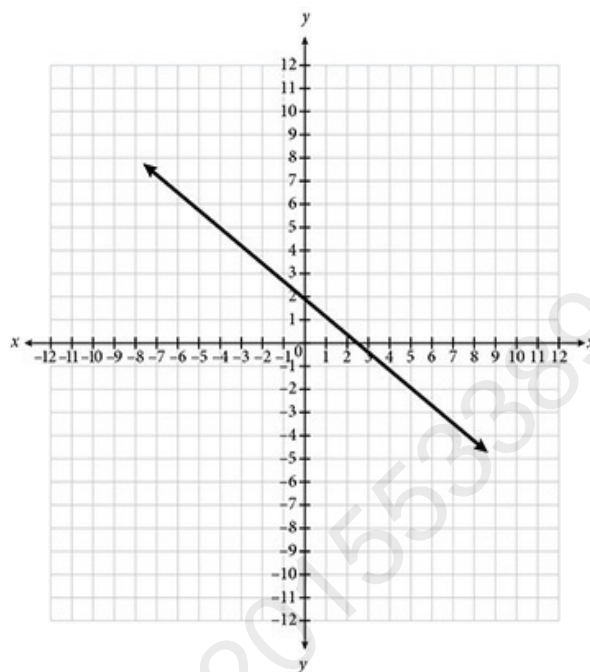
Vertical lines have no change in x . Since you cannot divide by zero, vertical lines have an undefined slope.

$$m = \frac{\text{change in } y}{0} = \text{undefined}$$

A line having *positive slope* rises from left to right, and a line having *negative slope* falls from left to right.



Positive slope



Negative slope

Example:

Find the slope of the line containing the points $S(6, -9)$ and $R(-2, -5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-5 - -9}{-2 - 6}$$

$$m = \frac{4}{-8}$$

$$m = \frac{-1}{2} \quad \text{Answer}$$

In the above example, we subtracted the coordinates of S from R . Order doesn't matter, however, as long as you choose the same order in the numerator and the denominator. We could have found the slope by subtracting the coordinates of R from S to get the same answer:

$$m = \frac{-9 - -5}{6 - -2} = \frac{-4}{8} = \frac{-1}{2}$$

Slope of Parallel and Perpendicular Lines

You can determine whether or not lines are parallel (never intersect) or perpendicular (intersect at a right angle) by examining their slope:

1. Parallel lines have the *same slope*: $m_1 = m_2$
2. Perpendicular lines have slopes that are *negative reciprocals*: $m_1 m_2 = -1$

Example:

Find the slope of the perpendicular bisector of \overline{AB} given $A(0, -2)$ and $B(3, 3)$.

The slope of \overline{AB} is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3 - -2}{3 - 0} = \frac{5}{3}$$

The slope of *any* line perpendicular to \overline{AB} , including the perpendicular bisector, is the opposite reciprocal of $\frac{5}{3}$.

$$m = \frac{-3}{5} \quad \text{Answer}$$

Equations of Lines

Linear equations are equations whose graphs are straight lines. Equations containing two variables x and y raised to the first power are linear. By definition, linear equations have a constant slope.

The following equations are NOT linear:

$$3x^2 + 2y^2 = 1 \qquad \frac{3}{x} + 4y = 5$$

The following equations are linear:

Slope-Intercept Form

$$y = mx + b$$

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Standard Form

$$Ax + By = C$$

The three forms of the equation of a line are further explained in the succeeding paragraphs.

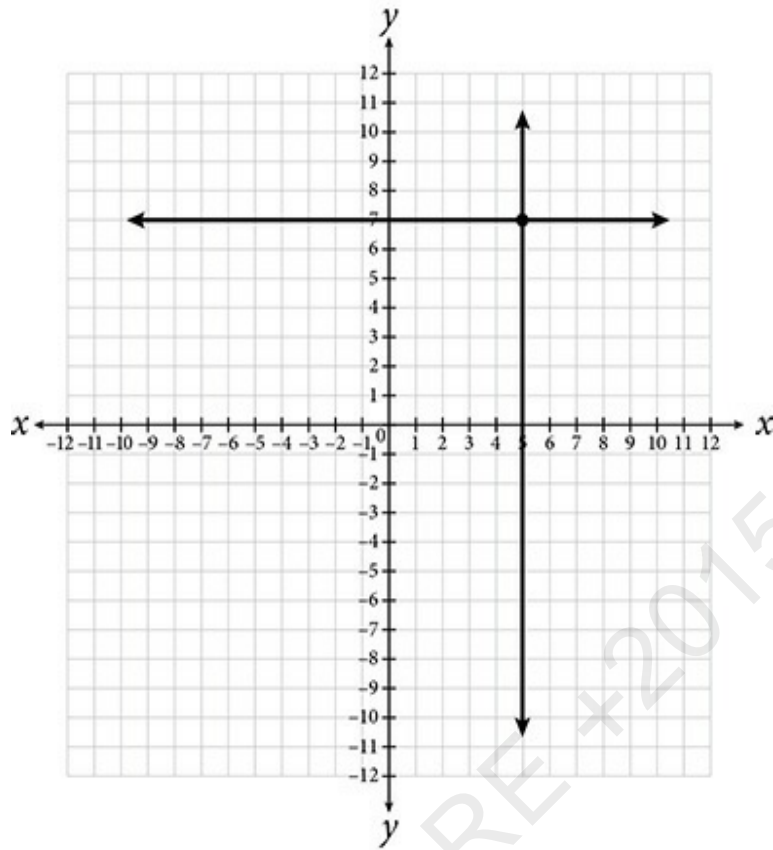
Horizontal and Vertical Lines

The most basic forms of linear equations are horizontal and vertical lines. Horizontal lines are written in the form $y = a$, where a is any constant. Vertical lines are in the form $x = a$, where, again, a is any constant.

Example:

Find the point of intersection of the lines $y = 7$ and $x = 5$.

The graph of the two lines clearly shows their intersection.



(5, 7) Answer

Standard Form

The *standard form* of the equation of a line is:

$$Ax + By = C \text{ (where } A \text{ and } B \text{ are both } \neq 0)$$

The *slope* of a line in standard form is $\frac{-A}{B}$ ($B \neq 0$).

Example:

Find the slope of the line $6x + 3y + 1 = 0$.

If you remember that the slope of a line in standard form is $\frac{-A}{B}$, this

problem can be quickly solved. You may want to rewrite the equation in standard form, although this does not affect the calculation of slope.

$$6x + 3y = -1$$

$$m = \frac{-6}{3} = -2 \quad \text{Answer}$$

Point-Slope Form

There is exactly one line passing through a given point having a given slope. Because this is true, a line can also be written in what is called point-slope form. The *point-slope form* of a line containing the point (x_1, y_1) with a slope of m is

$$y - y_1 = m(x - x_1)$$

Example:

Find the equation of the line in point-slope form through points $A(1, 10)$ and $B(3, 0)$.

Start by finding the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 10}{3 - 1} = \frac{-10}{2} = -5$$

Now, choose one of the two points and substitute its x - and y -coordinates into x_1 and y_1 in the point-slope form of a line. Let's choose point A :

$$y - 10 = -5(x - 1) \quad \text{Answer}$$

Choosing point B instead of A results in the same line. $y - 0 = -5(x - 3)$. You can compare the two equations by rewriting each in standard form:

$$y - 10 = -5(x - 1)$$

$$y - 10 = -5x + 5$$

$$5x + y = 15$$

$$y - 0 = -5(x - 3)$$

$$y = -5x + 15$$

$$5x + y = 15$$

Slope-Intercept Form

The *slope-intercept form* of a line with slope of m and y -intercept b is

$$y = mx + b$$

The y -intercept is the point where the line intersects the y -axis. If the y -intercept is 5, the line intersects the y -axis at the point $(0, 5)$. If the y -intercept is $\frac{-1}{2}$, the line intersects the y -axis at the point $\left(0, \frac{-1}{2}\right)$.

Example:

Which of the following lines is parallel to the line $y = 5x - 1$?

(A) $y = 5x - \frac{1}{5}$

(B) $y = -5x - \frac{1}{5}$

(C) $y = -5x + \frac{1}{5}$

(D) $y = \frac{-1}{5}x - 1$

(E) $y = \frac{-1}{5}x + 1$

Recall that parallel lines have the same slope. Answers B, C, D, and E

can quickly be eliminated, since their slopes do not equal 5, the x -coefficient of the given line. Answer A is the only equation in which $m = 5$, so A is the correct answer.

The correct answer is A. Answer

Example:

Find the equation of the line with a slope of 3 containing the point (9, 7). Write your answer in slope-intercept form.

Using $m = 3$, you can write the equation of the line as:

$$y = 3x + b$$

Since (9, 7) is a point on the line, let $x = 9$ and $y = 7$ and solve for b .

$$7 = 3(9) + b$$

$$7 = 27 + b$$

$$-20 = b$$

Now you have enough information to write the slope-intercept form of the line.

$$y = 3x - 20 \quad \text{Answer}$$

Since you are given the slope of the line and a point on it, you may be inclined to immediately use the point-slope form of the line. Doing so results in the equation:

$$y - 7 = 3(x - 9)$$

Now solve for y to get

$$y - 7 = 3x - 27$$

$$y = 3x - 20$$

Notice that you get the same final answer.

Determining x - and y -Intercepts

The *y*-intercept of a line is the point where $x = 0$ and the *x*-intercept of a line is the point where $y = 0$.

Example:

Determine the *x*- and *y*-intercepts of the line $3x - 4y = 12$.

Let's first determine the *y*-intercept by letting $x = 0$ and solving for *y*:

$$3(0) - 4y = 12$$

$$-4y = 12$$

$$y = -3$$

Now, let's determine the *x*-intercept by letting $y = 0$ and solving for *x*.

$$3x - 4(0) = 12$$

$$3x = 12$$

$$x = 4$$

The intercepts are $(0, -3)$ and $(4, 0)$. Answer

Example:

Find the equation of the line with *y*-intercept $(0, 4)$ and *x*-intercept $(-3, 0)$. Write your answer in standard form.

The slope of the line is $m = \frac{4 - 0}{0 - (-3)} = \frac{4}{3}$. Since its *y*-intercept is $(0,$

$4)$, $b = 4$.

In slope-intercept form, the equation of the line is

$$y = \frac{4}{3}x + 4$$

To determine the standard form of the equation, multiply through by 3.

$$3y = 4x + 12$$

$$-4x + 3y = 12 \quad \text{Answer}$$

Circles

So far, we have been focused on graphing lines in an xy -coordinate plane. The Level 1 test includes questions on some curved graphs; however, they mainly involve circles and parabolas. Circle questions involve manipulating the standard form of the equation of a circle.

Recall that a *circle* is defined as the set of all points at a given distance from a given point. The set of all points at a distance of 2 units from the origin can be written as:

$$x^2 + y^2 = 2^2$$

In standard form, the equation of a circle with center (h, k) and radius r is:

$$(x - h)^2 + (y - k)^2 = r^2$$

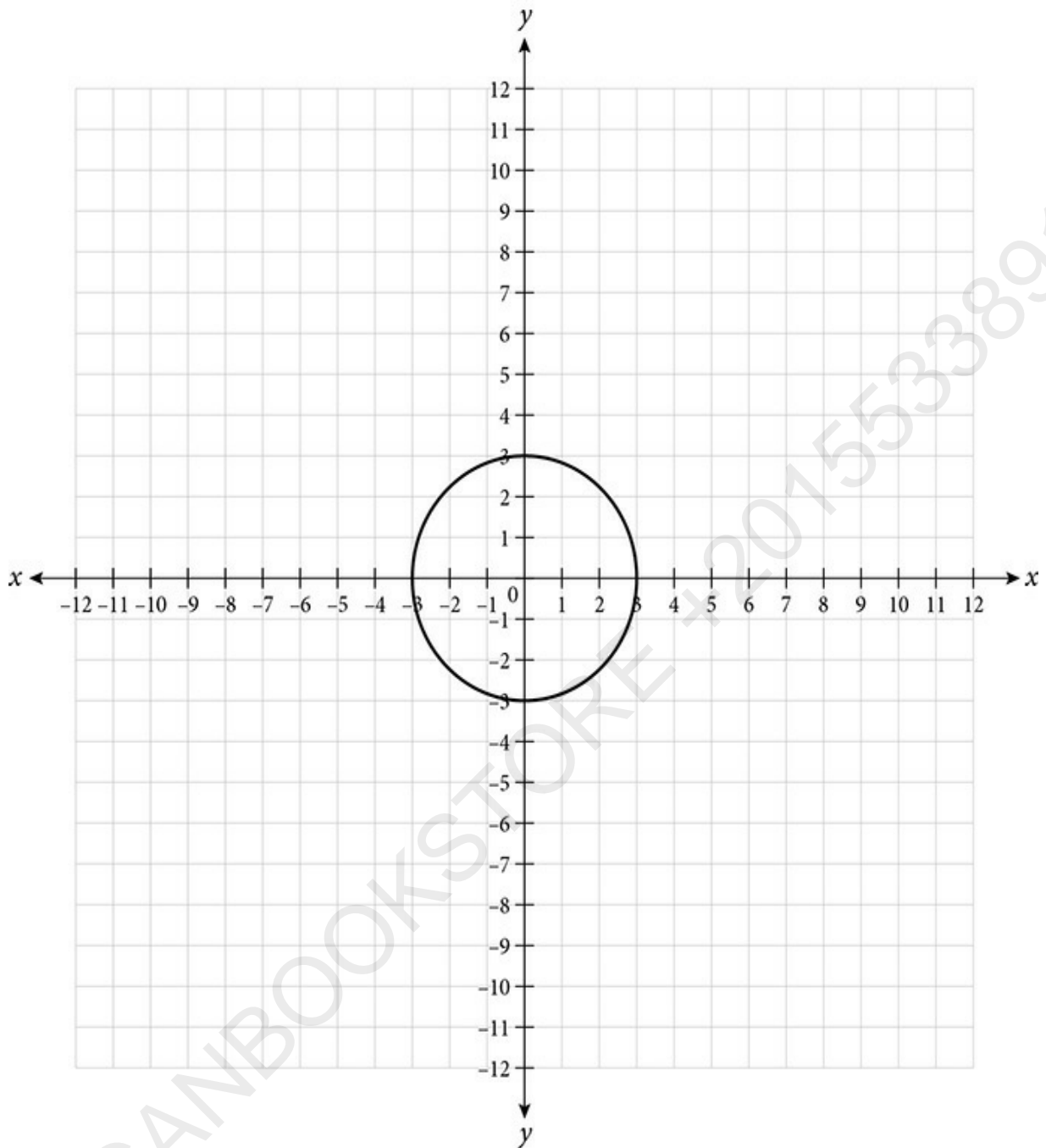
The set of all points at a distance of 5 from the point $(-2, 1)$ would, therefore, be

$$(x + 2)^2 + (y - 1)^2 = 5^2$$

Example:

Find the x - and y -intercepts of the graph of $x^2 + y^2 = 9$.

First, recognize that this is the graph of a circle with center $(0, 0)$ and radius 3. Picture the circle or sketch a graph to help visualize the intercepts.



This circle is the set of all points at a distance of 3 from the origin. The intercepts can be determined by moving 3 units to the right and left of the origin, and then moving 3 units up and down from the origin.

The intercepts are $(3, 0)$, $(-3, 0)$, $(0, 3)$, and $(0, -3)$. Answer

Example:

Find the equation of a circle with a center in quadrant II that is tangent to both the x - and y -axis with a radius of 4.

Since the circle's center is in quadrant II, its x -coordinate must be negative and its y -coordinate must be positive. The radius is 4, which tells you that the points $(0, 4)$ and $(-4, 0)$ are on the circle. These are the tangent points. The circle's center must, therefore, be $(-4, 4)$.

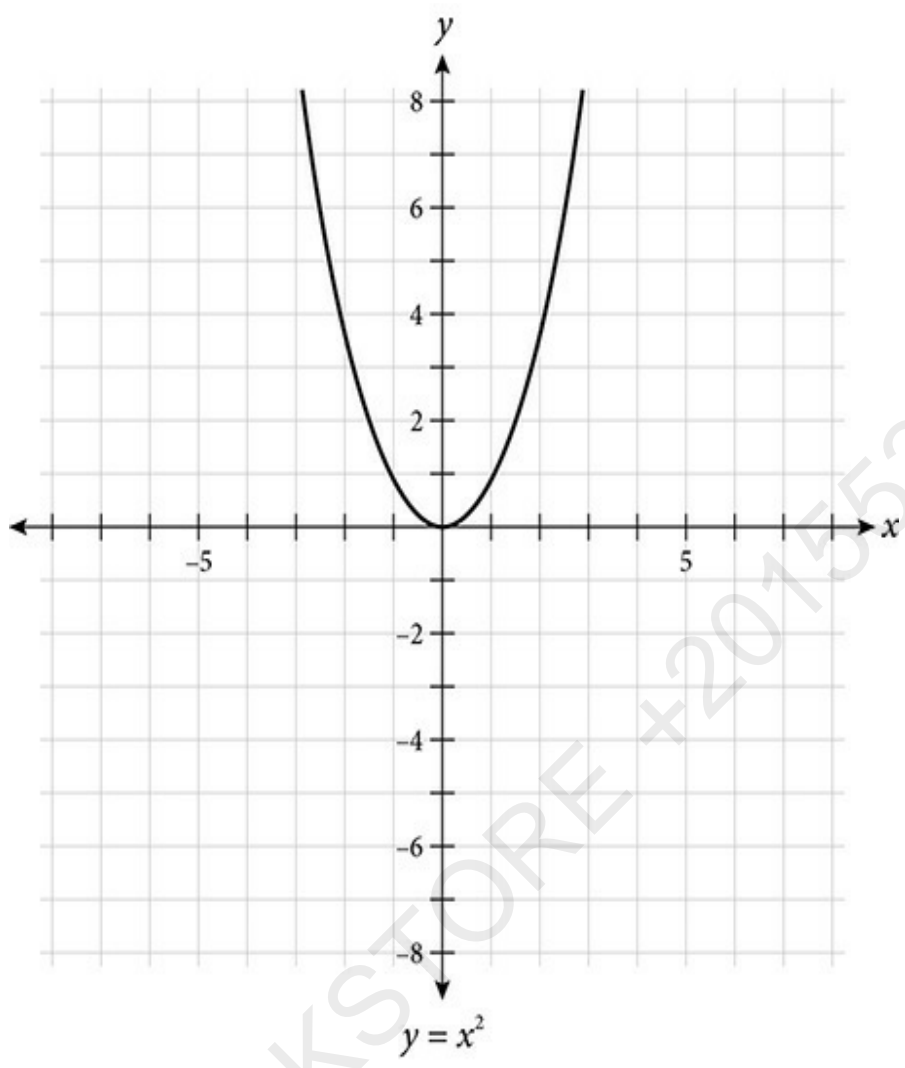
Now you have enough information to write the standard form of the equation of the circle:

$$(x - -4)^2 + (y - 4)^2 = 4^2$$

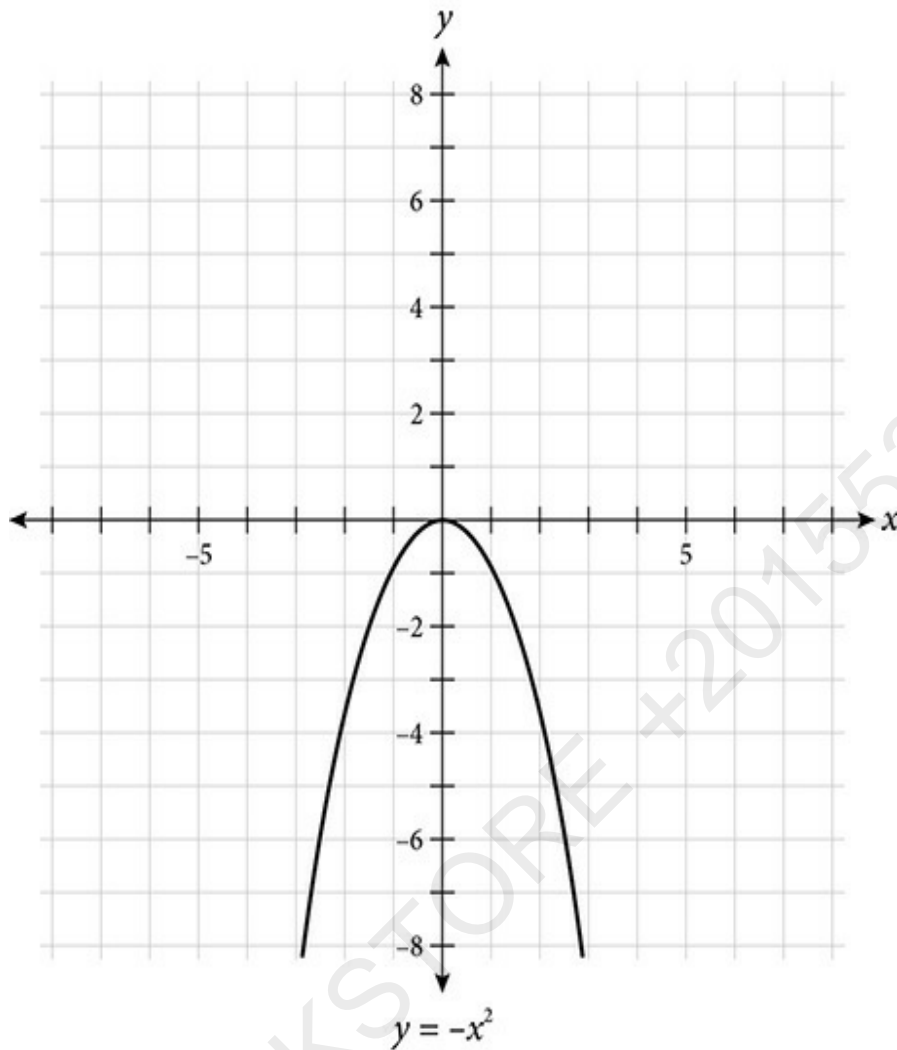
$$(x + 4)^2 + (y - 4)^2 = 16 \quad \text{Answer}$$

Parabolas

A *parabola* is the second type of curved graph that you may find on the Level 1 test. The *vertex* of a parabola is a turning point where a decreasing graph begins to increase, or vice versa. The most basic parabola is $y = x^2$. This parabola has a vertex at the origin and is concave up. Similarly, the graph of $y = -x^2$ also has a vertex at the origin but is concave down.



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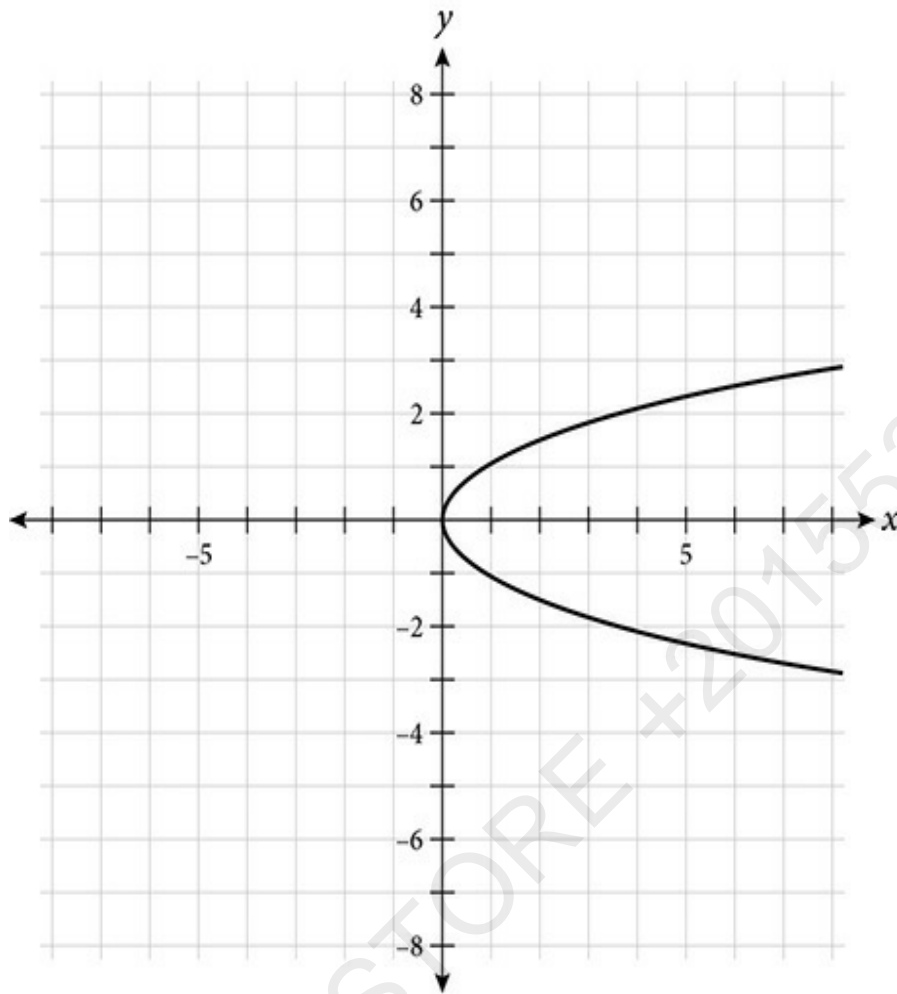
You should know that the graph of a quadratic equation $y = ax^2 + bx + c$ is a parabola. The standard form of a parabola is

$$y - k = a(x - h)^2$$

where (h, k) is the vertex.

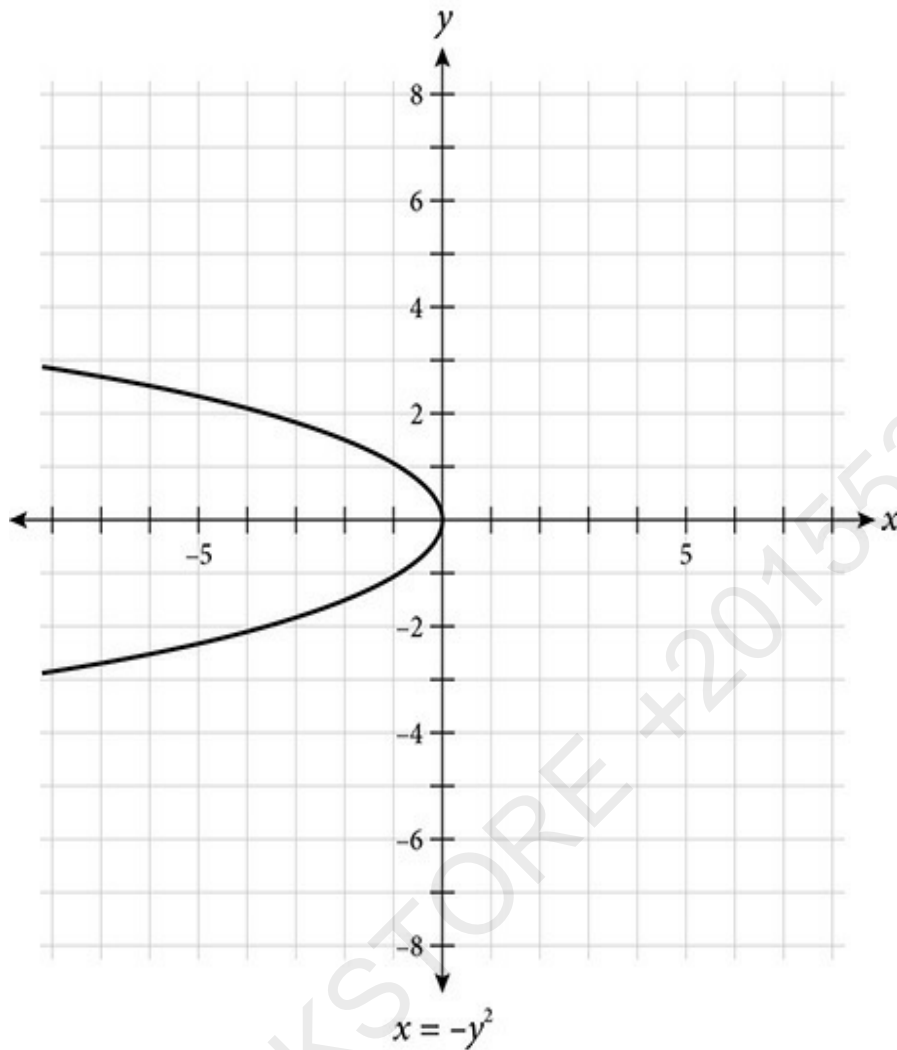
The graph opens upward when $a > 0$ and opens downward when $a < 0$. The greater $|a|$, the more narrow the graph becomes. Parabolas in this form have an axis of symmetry at $x = h$.

Parabolas can also be in the form $x - k = a(y - h)^2$. Parabolas in this form open right if $a > 0$, open left if $a < 0$, and have an axis of symmetry at $y = h$. The most basic parabolas in this form are $x = y^2$ and $x = -y^2$.



$$x = y^2$$

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Parabolas in the form $x - k = a(y - h)^2$ are NOT functions, though, since they do not pass the vertical line test. There is not a one-to-one correspondence between a given x -value and its corresponding y -values for these parabolas.

Example:

Determine the vertex of the graph of $y = (x + 1)^2 + 7$.

Recognize that this is the graph of a parabola. The parabola is concave up, since $a = 1$. In standard form, the equation becomes

$$y - 7 = (x + 1)^2$$

The vertex is $(-1, 7)$. Answer

As mentioned, the vertex of a parabola is a turning point where a decreasing graph begins to increase, or vice versa. Because of this, the vertex represents either the maximum or minimum value of the graph. The vertex is a *maximum* if the parabola is concave down and *minimum* if the parabola is concave up. The x -coordinate of the parabola is the x -value of the maximum or minimum of the function. The *maximum value of a function* is the y -coordinate of the vertex.

Example:

Find the minimum value of the function $f(x) = 2x + x^2$.

There are many ways to go about solving for the minimum of a function like this. Remember that the minimum value is the y -coordinate of the parabola's vertex. You could find the vertex by doing one of the following:

1. Complete the square to write the parabola in standard form.

$$y + 1 = x^2 + 2x + 1$$

$$y + 1 = (x + 1)^2$$

Since the vertex is $(-1, -1)$, the minimum value of the function is -1 .

2. Find its x -intercepts and determine its axis of symmetry.

$$y = x(2 + x)$$

$$0 = x(2 + x)$$

$x = 0$ or $x = -2$ so the intercepts are $(0, 0)$ and $(-2, 0)$.

The axis of symmetry is halfway between the two intercepts, so the axis is $x = -1$.

The x -coordinate of the vertex must be -1 , so solve for the y -value.

$$f(-1) = 2(-1) + (-1)^2 = -2 + 1 = -1$$

3. Use the fact that the maximum or minimum value of a quadratic

function occurs when $x = \frac{-b}{2a}$.

$$y = ax^2 + bx + c$$

$$f(x) = 2x + x^2, \text{ meaning that } a = 1 \text{ and } b = 2.$$

$$\frac{-b}{2a} = \frac{-2}{2(1)} = -1$$

Solve for y when $x = -1$ to get $f(x) = -1$.

4. Graph the parabola on your calculator to see where the vertex is.

The third solution above works for any parabolic function, $y = ax^2 + bx + c$. Remember that the maximum or minimum value of a function is the y -

value when $x = \frac{-b}{2a}$. The value is a maximum when $a < 0$ and a

minimum when $a > 0$.

Graphing Inequalities

Graphing inequalities results in the set of all ordered pairs (x, y) that make the inequality true when substituted for the variables. The set is usually infinite and is illustrated by a region in the plane. Think of an inequality as an equation. Graph the line represented by the equation (assuming the given equation is linear), and determine what region satisfies the inequality. Then shade this region.

Example:

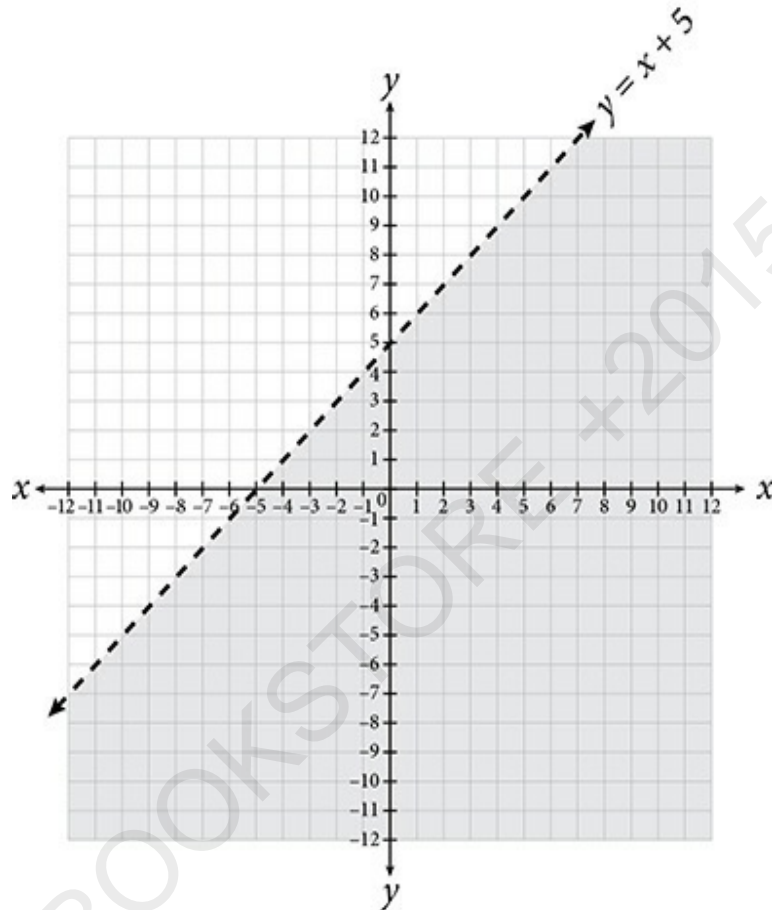
Graph $y < x + 5$.

Start by graphing the line $y = x + 5$. Since y is *less than* $x + 5$ and not *less than or equal to* $x + 5$, the line is dotted. Now determine whether to shade the region above or below the line. Choose a point below the line, say, $(0,$

2), and see if it satisfies the inequality.

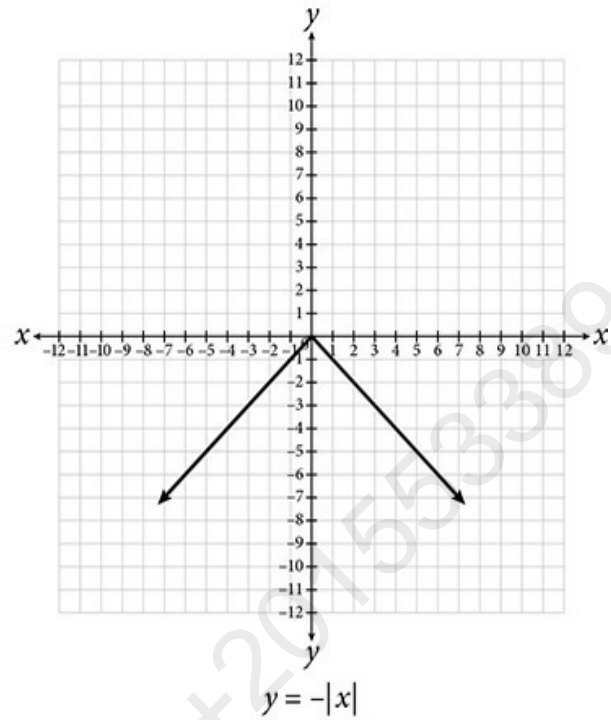
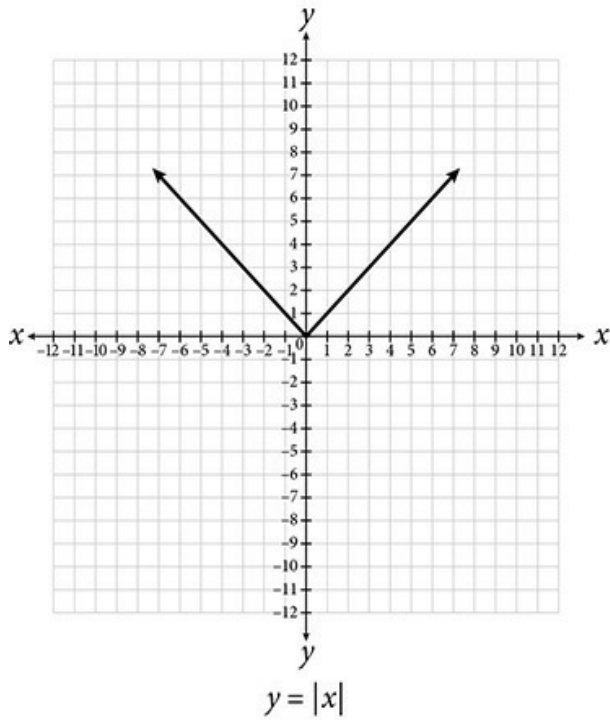
$$2 < 0 + 5$$

$2 < 5$ This is a true statement, so the region below the line $y = x + 5$ is the solution to the inequality. The graph is as follows:



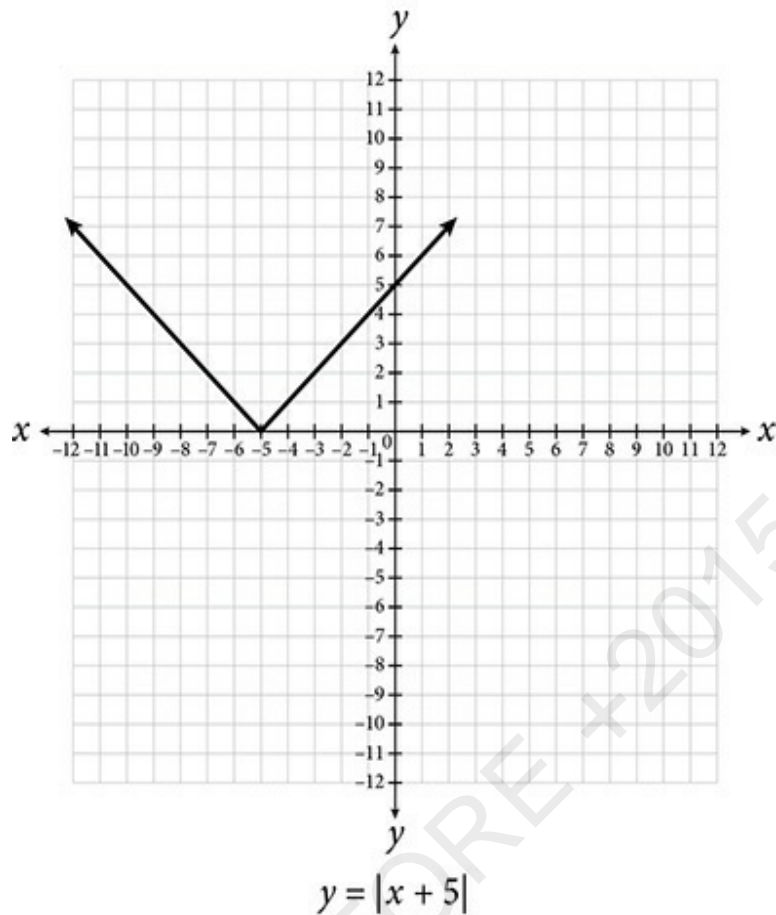
Graphing Absolute Value

Absolute value graphs are V-shaped. The most basic absolute value graphs are for $y = |x|$ and $y = -|x|$.



Example

Based on the previous example, try graphing $y = |x + 5|$. When $x = -5$, you get the turning point of the V-shaped graph. The graph is

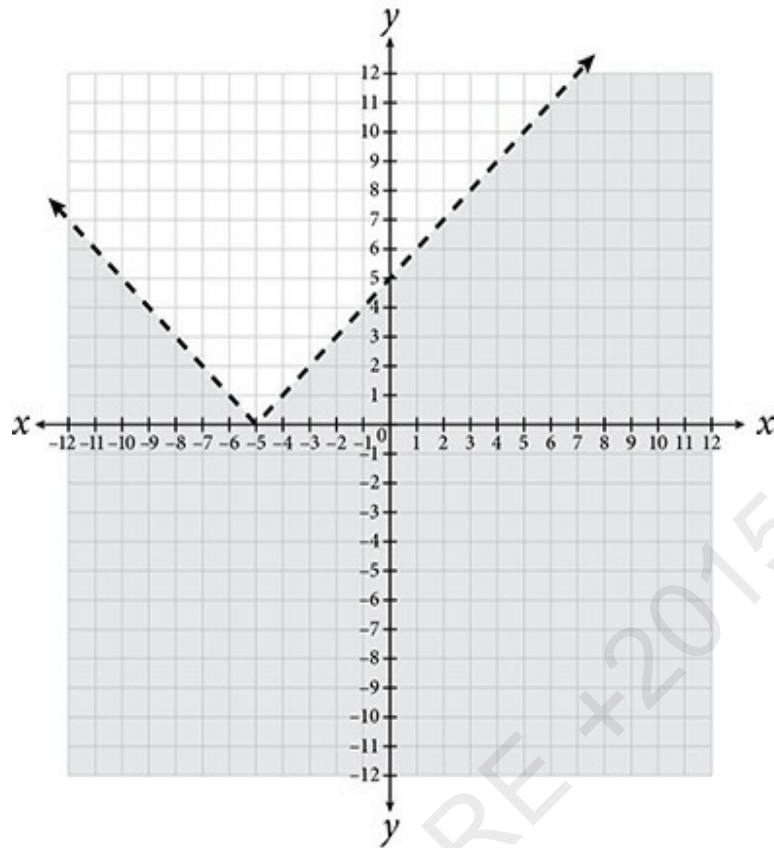


Now try graphing an absolute value inequality $y < |x + 5|$. The graph is similar to the graph of $y = |x + 5|$ with a dotted line at $y = |x + 5|$. As with any inequality, determine whether to shade above or below the line. Choose a point, say, $(-8, 1)$, and see if it satisfies the inequality.

$$1 < |-8 + 5|$$

$$1 < |-3|$$

$1 < 3$ This is a true statement, so the region below the graph satisfies the inequality. The graph is as follows:



$$y < |x + 5|$$

REVIEW QUESTIONS

1. What is the midpoint of the line segment with endpoints $(-2, 0)$ and $(3, -6)$?
 - (A) $(5, -6)$
 - (B) $(1, -6)$
 - (C) $\left(\frac{1}{2}, -3\right)$
 - (D) $\left(-4, \frac{3}{2}\right)$
 - (E) $\left(-3, \frac{1}{2}\right)$

2. Which of the following lines is perpendicular to the line $y = -\frac{1}{2}x + 3$?

(A) $-2x + 4$

(B) $\frac{1}{2}x + 3$

(C) $2x - 4$

(D) $-\frac{1}{2}x - 6$

(E) $2x + 4$

3. What is the vertex of a parabola $y = (x + 6)^2 - 4$?

(A) $(-36, -4)$

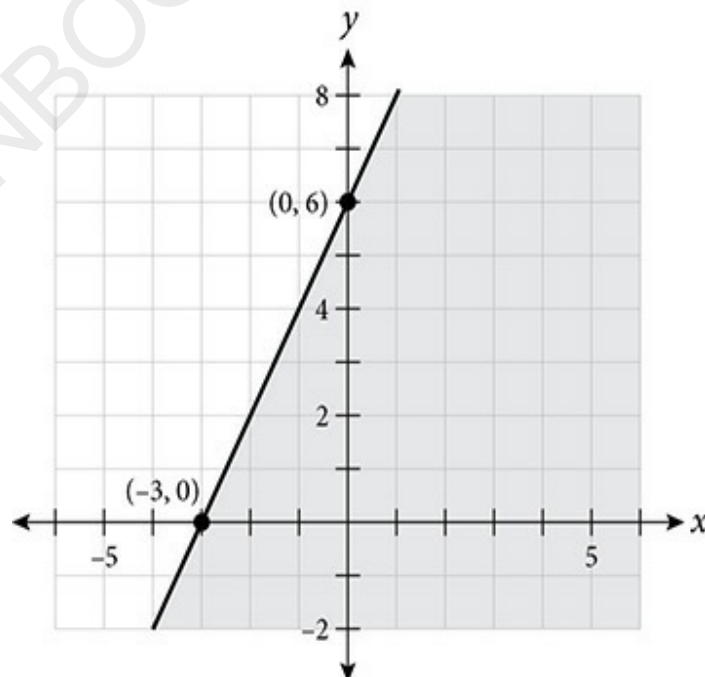
(B) $(-6, -4)$

(C) $(-4, -6)$

(D) $(4, 6)$

(E) $(6, 4)$

4. The graph shown is for



- (A) $y < 2x + 6$
- (B) $y \leq 2x + 6$
- (C) $y > 2x + 6$
- (D) $y \geq 2x + 6$
- (E) $y = 2x + 6$

5. What is the x -intercept for the line $8y + 3x - 5 = 0$?

- (A) $\left(\frac{5}{3}, 0\right)$
- (B) $\left(0, \frac{5}{3}\right)$
- (C) $\left(\frac{5}{2}, 0\right)$
- (D) $\left(0, \frac{5}{2}\right)$
- (E) $(3, 0)$

ANSWERS AND EXPLANATIONS

1. (C) The midpoint is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ where (x_1, y_1) and (x_2, y_2) are the endpoints. In this case, $(-2, 0)$ and $(3, -6)$.

$$\left(\frac{-2+3}{2}, \frac{0+(-6)}{2}\right) = \left(\frac{1}{2}, \frac{-6}{2}\right) = \left(\frac{1}{2}, -3\right)$$

2. (E) The slopes of two perpendicular lines are negative reciprocals of each

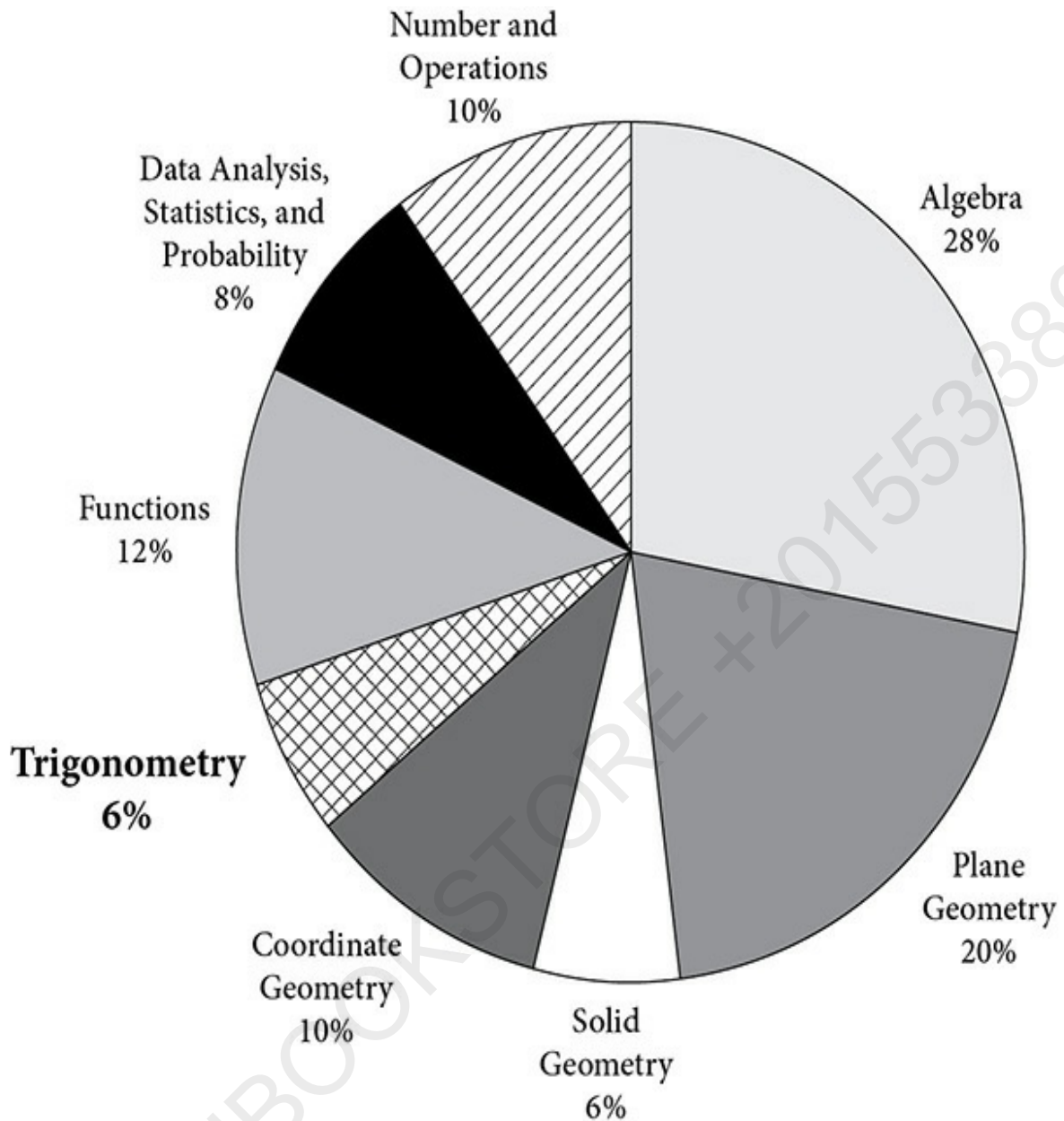
other, so a line with slope of $-\frac{1}{2}$ will have a negative reciprocal slope of 2 for a line perpendicular to it.

3. **(B)** Put the equation in standard form: $y - k = a(x - h)^2$ $y + 4 = (x + 6)^2$. The vertex is (h, k) , which in this case is $(-6, -4)$.
4. **(B)** Since the graph has a portion shaded, it is an inequality. Since the line is solid, it is a greater-than-or-equal-to graph or a less-than-or-equal-to graph. Use the answer choices with the graph to decide which. Pick a point within the shaded area, such as $(2, 2)$. Use those x - and y -coordinates to see whether choice B or D is true. For B, $2 \leq 2(2) + 6$ is true, so the correct answer is B.
5. **(A)** Plug in 0 for y and solve. $2(0) + 3x - 5 = 0$; $0 + 3x - 5 = 0$; $3x = 5$; $x = \frac{5}{3}$. The x -intercept is $\left(\frac{5}{3}, 0\right)$.



Trigonometry

This chapter provides a review of basic trigonometry principles. Six to eight percent of the Level 1 test questions relate to trigonometry. The six to eight percent works out to be three or four questions, which are usually grouped with the more difficult questions toward the end of the Level 1 test. The pie chart shows approximately how much of the Level 1 test is related to trigonometry:

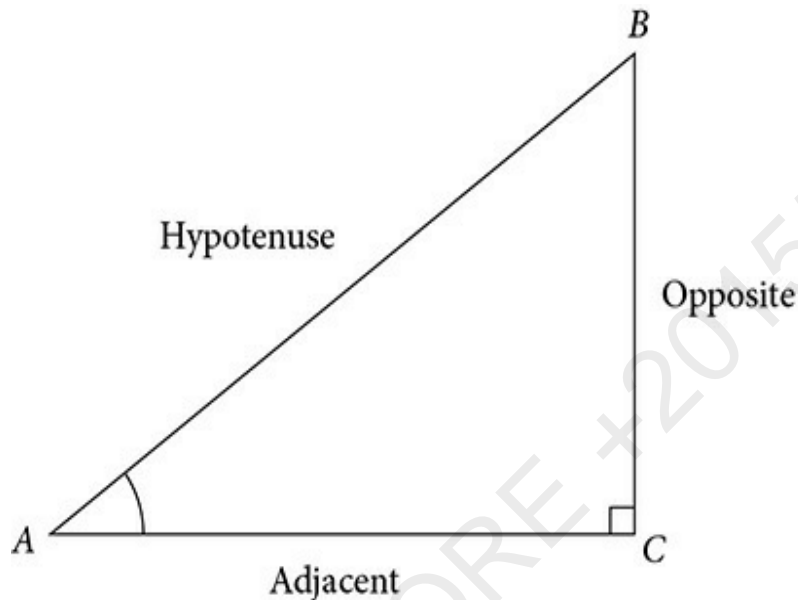


The following trigonometry topics are covered in this chapter:

1. Right Triangle Trigonometry
2. Inverse Functions
3. Special Right Triangles
4. Trigonometric Identities

Right Triangle Trigonometry

Trigonometry means “the measurement of triangles,” and one aspect of trigonometry relates to the study of the relationships between the sides and angles of right triangles. Those three ratios are presented below. They are true when focusing on one acute angle in a right triangle. Take $\triangle ABC$, for example:



\overline{BC} is the side opposite the given angle (in this case $\angle A$). \overline{AC} is the side adjacent, or next to, $\angle A$. \overline{AB} is the hypotenuse, since it's the side opposite the 90° angle. The three trigonometric ratios for $\angle A$ are

$$\text{sine } \angle A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\overline{BC}}{\overline{AB}}$$

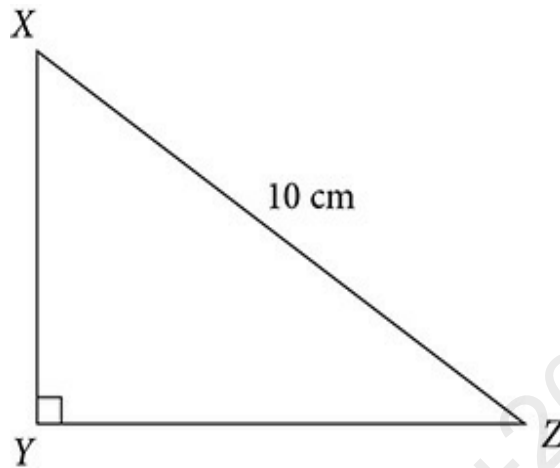
$$\text{cosine } \angle A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\overline{AC}}{\overline{AB}}$$

$$\text{tangent } \angle A = \frac{\text{opposite}}{\text{adjacent}} = \frac{\overline{BC}}{\overline{AC}}$$

The abbreviations SOH, CAH, and TOA are used to remember the ratios shown above. Typically sine, cosine, and tangent are abbreviated as *sin*, *cos*, and *tan*, as they are on the your calculator's buttons.

Example:

In the right triangle below, $m\angle XZY = 35^\circ$. Find the length of \overline{XY} . Round your answer to the nearest hundredth.



You know the length of the hypotenuse and you're missing the length of XY , the side opposite $\angle XZY$. The *sine ratio* will enable you to determine

XY , since $\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$.

$$\sin 35^\circ = \frac{\overline{XY}}{10}$$

$$10(\sin 35^\circ) = \overline{XY}$$

$$10(0.57358) = \overline{XY}$$

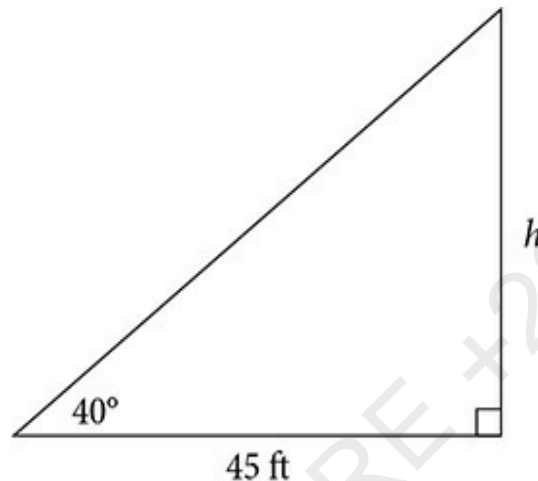
$$\overline{XY} = 5.74 \text{ cm} \quad \text{Answer}$$

Make sure your calculator is set to *degree mode* to do the trigonometry problems on the Level 1 test, because all angles will be in degrees. (Angle measures on the Level 2 test may, however, be given in radians.) To check the setting of your calculator, determine $\sin 30^\circ$. In degree mode, $\sin 30^\circ = 0.5$. If your calculator is incorrectly set to radian mode, $\sin 30^\circ = -0.98803$. Change your mode to degree if this is the case.

Example:

At a distance of 45 feet from a flagpole, the angle from the ground to the top of the flagpole is 40° . Find the height of the flagpole. Round your answer to the nearest hundredth.

It may help to sketch a diagram to solve this problem.



Now, determine which trigonometric ratio could be used to solve for h . You don't have enough information to use sine or cosine, since you don't know the length of the hypotenuse. Tangent = $\frac{\text{opposite}}{\text{adjacent}}$, so

$$\tan 40^\circ = \frac{h}{45}$$

$$45(\tan 40^\circ) = h$$

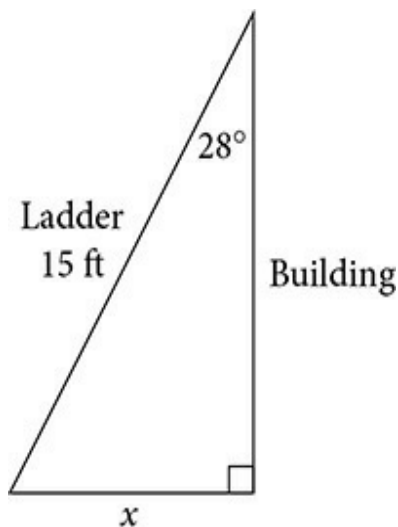
$$45(0.83910) = h$$

$$h = 37.76 \text{ feet} \quad \text{Answer}$$

Example:

A 15-foot ladder leans against the side of a building, creating an angle of 28° with the building. How far is the base of the ladder from the side of

the building?



The ladder problem is commonly used in right triangle trigonometry problems. First, determine which trigonometric ratio to use. Since you know the hypotenuse and are trying to solve for the side opposite the 28° angle, it makes sense to use sine.

$$\sin 28^\circ = \frac{x}{15}$$

$$15(\sin 28^\circ) = x$$

$$x = 7.04 \text{ feet} \quad \text{Answer}$$

Inverse Functions

Trigonometry can also be used to determine the measures of angles. $\sin^{-1} x$ denotes the inverse sine of x , or the *arcsine*.

$\sin^{-1} (0.707)$ means “the angle whose sine is 0.707.”

To solve for the angle measure, use the \sin^{-1} function on your calculator. (On most graphing calculators, this is accessed by pressing the second button and then the sin button.) Try calculating $\sin^{-1} (0.707)$ using your calculator. It

should equal approximately 45° .

Similarly, $\cos^{-1} x$ denotes the inverse cosine of x , or the *arccosine*, and $\tan^{-1} x$ denotes the inverse tangent of x , or the *arctangent*. Both inverse functions can be solved in the same manner as arcsine by using the \cos^{-1} and \tan^{-1} function on your calculator.

Using your calculator, find the value of each of the following:

1. $\sin^{-1}(0.5)$ (You're solving for "the angle whose sine is 0.5.")
2. $\cos^{-1}(0.8)$ (You're solving for "the angle whose cosine is 0.8.")
3. $\tan^{-1}(1)$ (You're solving for "the angle whose tangent is 1.")

The answers are, in order, 30° , 36.9° , and 45° . If you didn't get these solutions, check to make sure your calculator is set in degree mode.

Example:

Given $\triangle JET$ is a 3-4-5 right triangle with a right angle at vertex E , determine the measures of the two acute angles. Round answers to the nearest tenth.

Let's use arcsine to solve for both angles. Remember that

$\sin = \frac{\text{opposite}}{\text{hypotenuse}}$ and the hypotenuse of $\triangle JET$ is its longest side. The

sine of one of the acute angles is, therefore, $\frac{3}{5}$, while the sine of the

second acute angle is $\frac{4}{5}$. Write two expressions using arcsine:

$$\sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}(0.6)$$

$$= 36.9^\circ$$

$$\sin^{-1}\left(\frac{4}{5}\right)$$

$$= \sin^{-1}(0.8)$$

$$= 53.1^\circ$$

The two acute angles measure 36.9° and 53.1° . Answer

Example:

The diagonals of a rhombus measure 12 and 16 inches. What is the measure of the larger angle of the rhombus?

Recall that the diagonals of a rhombus are perpendicular, bisect each other, bisect the vertex angles of the rhombus, and form four congruent right triangles. Let x and y equal the measures of the acute angles of the right triangles.

Let x = the measure of the angle adjacent to the side measuring 8 units.

Let y = the measure of the angle adjacent to the side measuring 6 units.

$$\tan x = \frac{6}{8}$$

$$x = \tan^{-1} \frac{6}{8} = 36.87^\circ$$

$$\tan y = \frac{8}{6}$$

$$y = \tan^{-1} \frac{8}{6} = 53.13^\circ$$

One angle of the rhombus measures $2(36.87) \approx 73.7^\circ$, while the other angle measures $2(53.13) \approx 106.3^\circ$. Since the problem asks for the greater of the two angles, 106.3° is the correct answer.

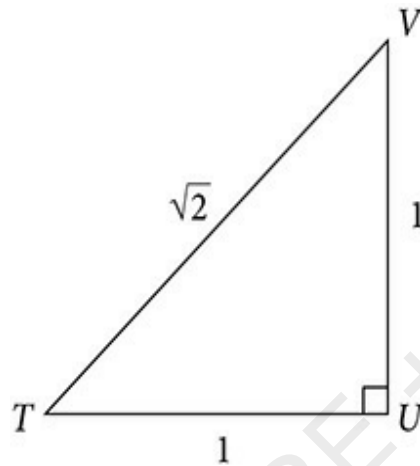
106.3° Answer

Special Right Triangles

Recall the two special right triangles mentioned in the Plane Geometry chapter, 45° - 45° - 90° and 30° - 60° - 90° triangles. Here they are mentioned again, because the ratios of their sides occur often in right triangle

trigonometry. A $45^\circ-45^\circ-90^\circ$ triangle is an isosceles right triangle whose sides are in the ratio $x:x:x\sqrt{2}$, while a $30^\circ-60^\circ-90^\circ$ triangle is a scalene right triangle whose sides are in the ratio $x:x\sqrt{3}:2x$.

For each special right triangle below, identify the sine, cosine, and tangent of the acute angles:



$$\begin{aligned}\sin \angle T &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

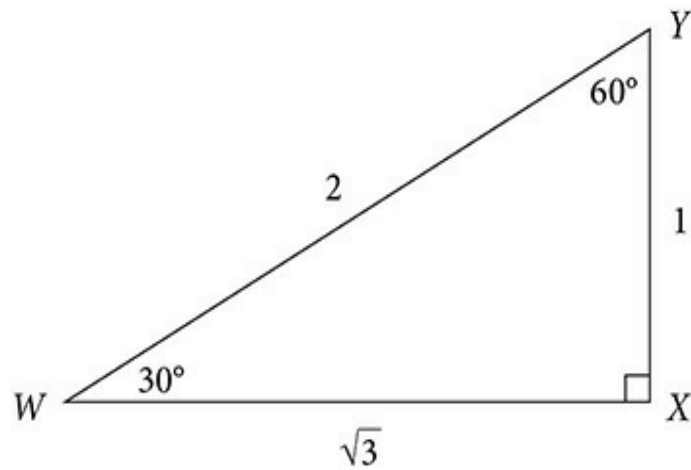
$$\begin{aligned}\cos \angle T &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\tan \angle T = \frac{1}{1} = 1$$

$$\begin{aligned}\sin \angle V &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\cos \angle V &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\tan \angle V = \frac{1}{1} = 1$$



$$\sin \angle W = \frac{1}{2}$$

$$\cos \angle W = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} \tan \angle W &= \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}}{\sqrt{3}} \right) \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

$$\sin \angle Y = \frac{\sqrt{3}}{2}$$

$$\cos \angle Y = \frac{1}{2}$$

$$\tan \angle Y = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Trigonometric Identities

You may encounter problems involving one key trigonometric identity on the Level 1 test:

$$\sin^2 x + \cos^2 x = 1$$

An identity is an equation true for every value of the domain. Try substituting any number in for x and see what happens. Let's try $x = 25^\circ$.

$$\sin^2(25^\circ) + \cos^2(25^\circ) = 1$$

$$0.422618^2 + 0.906308^2 = 1$$

Example:

Simplify $(4\sin x)(3\sin x) - (\cos x)(-12\cos x)$.

First, multiply the terms:

$$12\sin^2x - -12\cos^2x$$

Notice that this resembles the identity $\sin^2x + \cos^2x = 1$. Now, simply factor out a 12.

$$= 12(\sin^2x + \cos^2x)$$

$$= 12(1)$$

$$= 12 \quad \text{Answer}$$

Example:

Simplify $(1 + \sin \alpha)(1 - \sin \alpha)$.

Recall that $\sin^2 \alpha + \cos^2 \alpha = 1$, so $\cos^2 \alpha = 1 - \sin^2 \alpha$.

$$(1 + \sin \alpha)(1 - \sin \alpha) =$$

$$1 - \sin \alpha + \sin \alpha - \sin^2 \alpha =$$

$$1 - \sin^2 \alpha =$$

$$\cos^2 \alpha \quad \text{Answer}$$

REVIEW QUESTIONS

1. In [Figure 1](#), what is the cosine of $\angle C$?

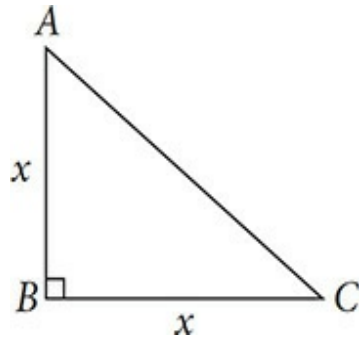


Figure 1

- (A) 1
- (B) $\frac{\sqrt{2}}{3}$
- (C) $\frac{\sqrt{2}}{2}$
- (D) $\frac{\sqrt{3}}{3}$
- (E) $\sqrt{2}$

2. In Figure 2, what is the sum of \overline{AB} and \overline{BC} ?

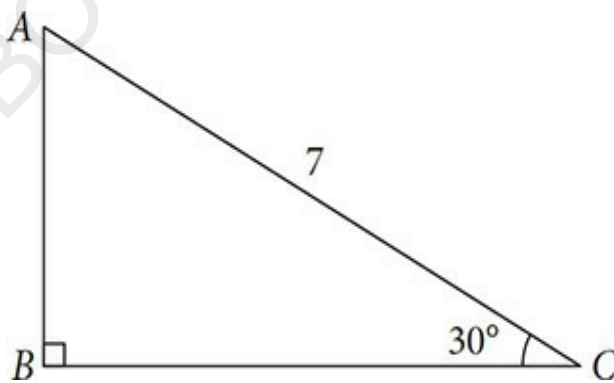


Figure 2

- (A) 0.87
- (B) 2.56
- (C) 3.50

- (D) 6.06
- (E) 9.56

3. If $0^\circ < x < 90^\circ$ and $\sin x = \frac{1}{3}$, what is $\cos x - \tan x$?

- (A) 2.83
- (B) 1.29
- (C) 0.94
- (D) 0.59
- (E) 0.35

4. In Figure 3, what is the tangent of $\angle A$?

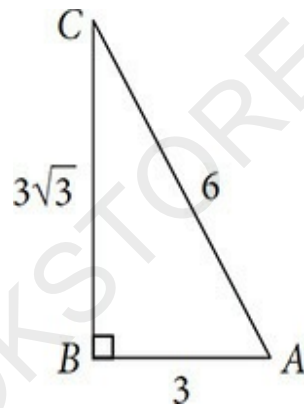


Figure 3

- (A) 1.73
- (B) 1.16
- (C) 0.87
- (D) 0.58
- (E) 0.50

5. In Figure 4, what is the sum of $\sin x$ and $\tan y$?

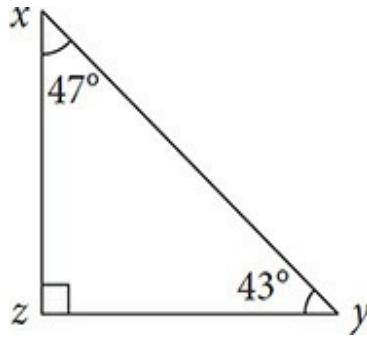


Figure 4

- (A) 0.84
- (B) 0.93
- (C) 1.54
- (D) 1.66
- (E) 2.22

ANSWERS AND EXPLANATIONS

1. (C) Since this is an isosceles right triangle, you know $\angle A = \angle C = 45^\circ$, but since the answer choices are not given in decimal form, your calculator may not be much help.

$\cos C = \frac{\text{adjacent}}{\text{hypotenuse}}$ and in an isosceles right triangle, the sides are in

the ratio $x : x : x\sqrt{2}$, so the hypotenuse $= x\sqrt{2}$.

$$\cos C = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) = \frac{\sqrt{2}}{2}$$

2. (E) To find \overline{AB} , use the sine ratio: $\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}}$

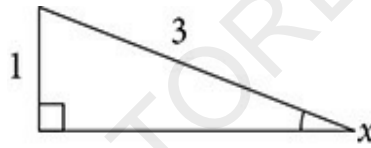
$$\sin 30^\circ = \frac{\overline{AB}}{7}; \overline{AB} = (\sin 30^\circ)(7) = (.5)(7) = 3.5$$

To find \overline{BC} , use the cosine ratio: $\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}}$

$$\cos 30^\circ = \frac{\overline{BC}}{7}; \overline{BC} = (\cos 30^\circ)(7) = (.866)(7) = 6.062$$

Add \overline{AB} and \overline{BC} : $3.5 + 6.062 = 9.562 = 9.56$

3. (D) sine = $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{3}$. Draw the triangle.



Use the Pythagorean Theorem to find the adjacent side.

$$1^2 + a^2 = 3^2$$

$$1 + a^2 = 9$$

$$a^2 = 8$$

$$a = \sqrt{8}$$

$$\text{cosine } x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{8}}{3}$$

$$\text{tangent } x = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{8}}$$

$$\cos x - \tan x = \frac{\sqrt{8}}{3} - \frac{1}{\sqrt{8}} = 0.94 - 0.35 = 0.59$$

4. (A) $\text{tangent} = \frac{\text{opposite}}{\text{adjacent}} = \frac{3\sqrt{3}}{3} = \sqrt{3} = 1.73$

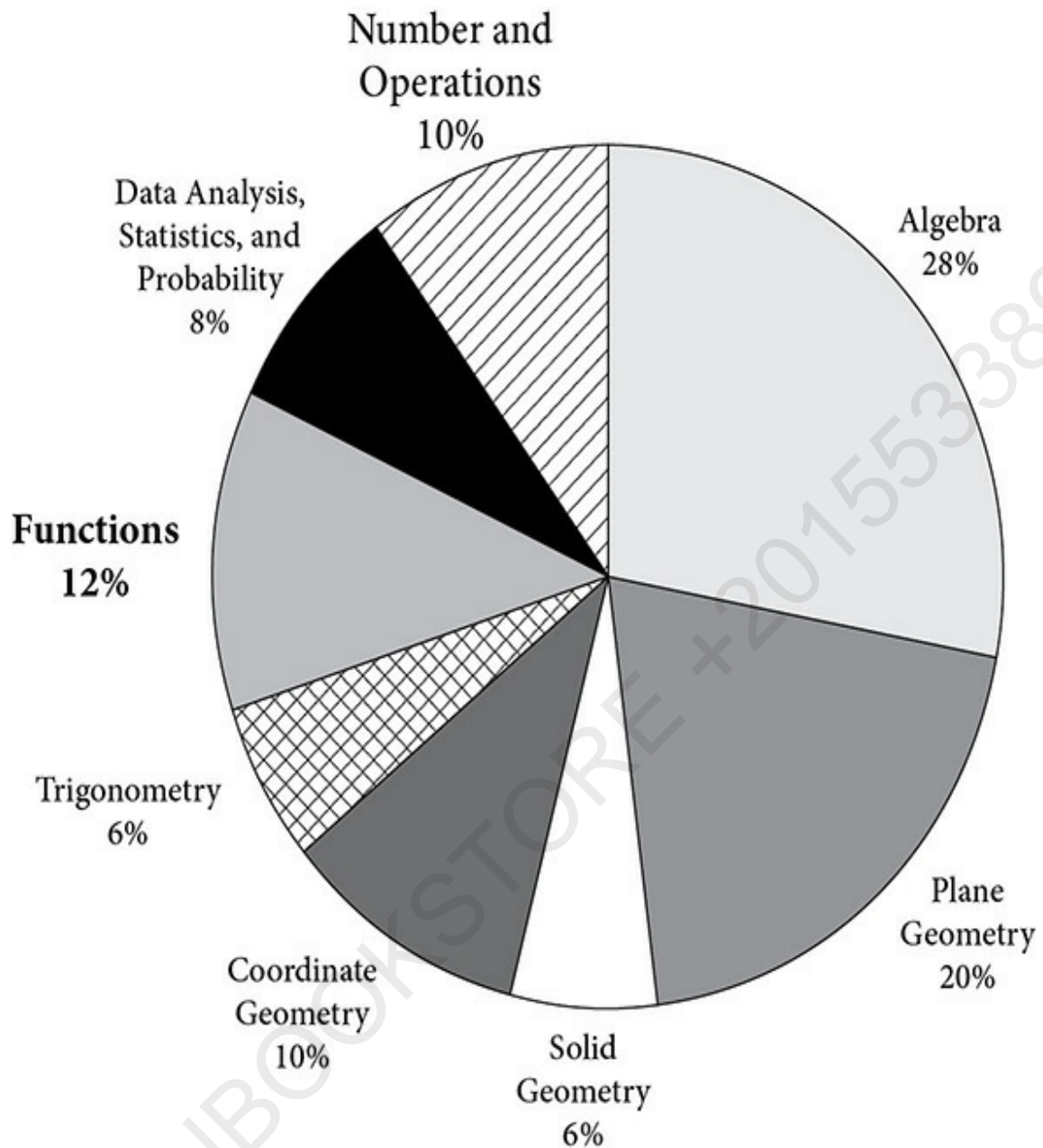
5. (D) $\sin 47^\circ + \tan 43^\circ = 0.73 + 0.93 = 1.66$

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Functions

This chapter provides a review of functions, focusing on algebraic functions. Thirty eight to 42 percent of the Level 1 test questions relate to Algebra and Functions combined. That translates to about 12 percent of the Level 1 test questions relating specifically to functions. Logarithmic, piecewise, recursive, parametric, and trigonometric functions are topics on the Level 2 test and are not covered here. The pie chart shows approximately how much of the Level 1 test is related to functions:



The following topics are covered in this chapter:

1. Functional Notation
2. Functions vs. Relations
 - a. Graphing Functions
3. Composition of Functions
 - a. Identity, Zero, and Constant Functions
4. Determining the Maximum or Minimum
5. The Roots of a Quadratic Function

6. Inverse Functions
7. Rational Functions
8. Higher-Degree Polynomial Functions
9. Exponential Functions

Functional Notation

A *function* is a set of ordered pairs of real numbers. A function is often described by a rule that creates a one-to-one correspondence between sets of input and output values. The set of all input values is called the *domain* of the function, and the set of all output values is called the *range*. Unless otherwise noted the domain and range are assumed to be the set of all real numbers. Typically, a function is denoted by $f(x)$ and is read “ f of x ” or “the value of the function f at x .” Functions can be represented by any letter, not just f , though. It is common to use f , F , g , G , h , and H to represent different functions. The following are examples of functions:

$$f(x) = x^2$$

$$F(x) = 3x$$

$$g(x) = (x - 4)^2 + 1$$

Take the first example, $f(x) = x^2$. When the function is evaluated for $x = 2$, the result is $f(2) = 2^2 = 4$. $f(x)$ represents the y -value when a function is graphed on an xy -coordinate plane. The point $(2, 4)$ would be included in the graph of the function $f(x) = x^2$ because when $x = 2$, $f(x) = 4$.

For a set of ordered pairs to be a function:

1. For each domain value of a function, there *must* be an associated range value.
2. A domain value cannot be matched with more than one range value.

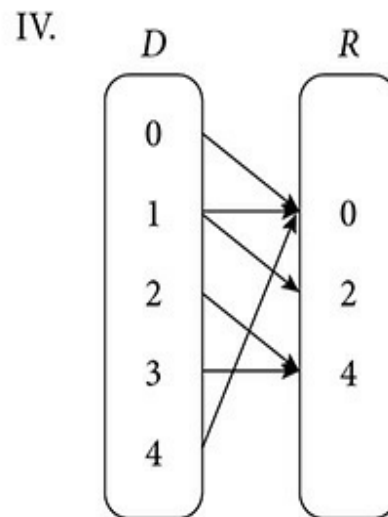
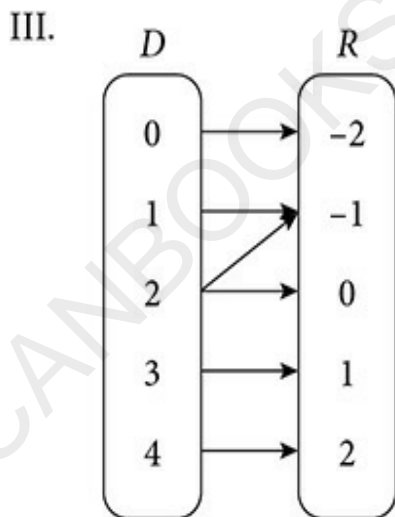
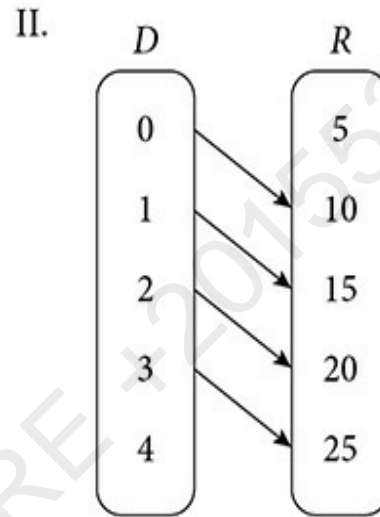
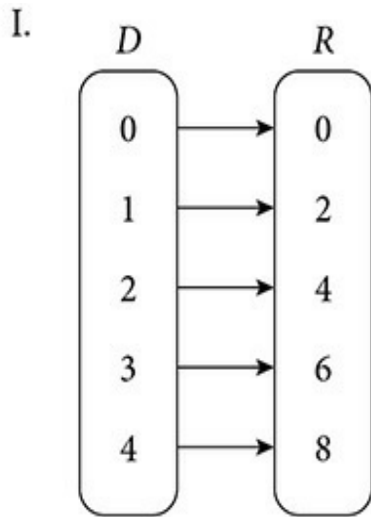
When determining the domain of a function, look for two common restrictions: dividing by zero and identifying when the radicand is negative. You want to ensure that both cases do not happen.

A *linear function* is a function in the form $f(x) = mx + b$, where x , m , and b are real numbers. The graph of a linear function is a straight line with slope m and y -intercept b .

A quadratic function is a function in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of a quadratic function is a *parabola*.

Example:

Which of the following mapping diagrams depicts a function?



- (A) I and II
- (B) II and III
- (C) III and IV
- (D) I only
- (E) II and IV

Remember that there must be a range associated with each domain value. II is not a function because 4 doesn't have an associated range. III cannot be a function because 2 is matched with both -1 and 0 . IV cannot be a function because 1 is matched with both 0 and 2 . I is the only function. D is the correct answer.

D Answer

Example:

Find the domain of $f(x) = \sqrt{100 - x^2}$.

The *domain* is the set of x -values for which the function is defined. In other words, it's the set of values allowed for x . We are assuming that the domain of the function is the set of real numbers. (you can assume this unless told otherwise.) The *radicand*, $100 - x^2$, must therefore be positive.

$$100 - x^2 \geq 0$$

$$100 \geq x^2$$

$$10 \geq x \text{ and } 10 \geq -x$$

$$10 \geq x \text{ and } -10 \leq x$$

$$-10 \leq x \leq 10 \quad \text{Answer}$$

Example:

Given $f(x) = 2x - 5$, find $f(x - 3)$.

Replace x in the original function with the expression $x - 3$.

$$f(x - 3) = 2(x - 3) - 5$$

$$f(x - 3) = 2x - 6 - 5$$

$$f(x - 3) = 2x - 11 \quad \text{Answer}$$

Example:

Determine the domain of the function $g(x) = \frac{1}{x(x+3)}$.

The fraction $\frac{1}{x(x+3)}$ is not defined when the denominator $x(x+3)$ equals zero. Dividing by zero is a common restriction of the domain of a function.

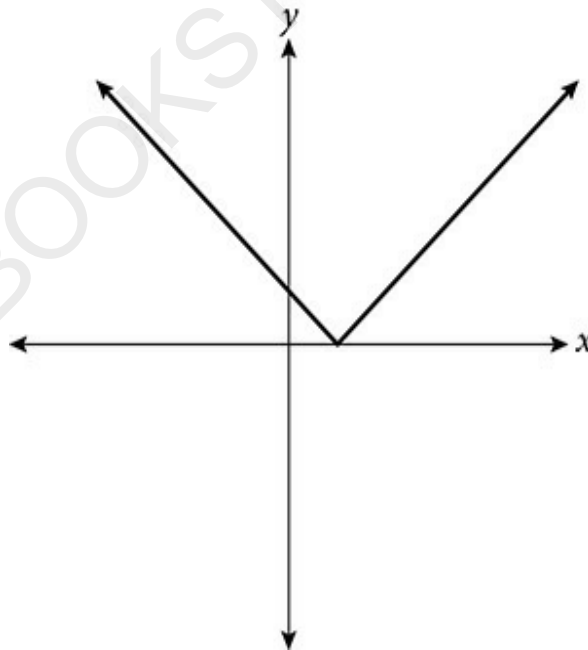
$$x(x+3) \neq 0$$

$$x \neq 0 \text{ or } x \neq -3$$

The domain is the set of all real numbers except 0 and -3. Answer

Example:

Find the domain and range of the function shown in the graph:



Although it is not asked in this question, do you know the equation associated with this graph? It is useful to know that “V-shaped” graphs are associated with absolute value, and this is the graph of $y = |x - 1|$. The

domain is the set of x for which the function is defined. Are there any restrictions on x in this example? The graph shows that x is defined at zero and at all positive and negative values. The domain is the set of all real numbers. What about the range? Looking at the equation, you see that $y = |x - 1|$. Since y equals the absolute value of an expression and absolute value, by definition, is always positive, y must be greater than or equal to zero. You can determine this from the graph since there are no points in either quadrant III or IV where y is negative.

$$D = \{\text{all real numbers}\}, R = \{y: y \geq 0\} \quad \text{Answer}$$

Functions vs. Relations

You can test for a function algebraically or by using the Vertical Line Test. The *Vertical Line Test* states that any vertical line intersects the graph of a function at, at most, one point. Both tests (algebraic and Vertical Line) determine if there is one and only one range (y value) associated with a given domain (x value).

Example:

Is $y^2 = 5 + x$ a function?

Let's test this algebraically. Start by solving for y .

$$y^2 = 5 + x$$

$$y = \pm \sqrt{5 + x}$$

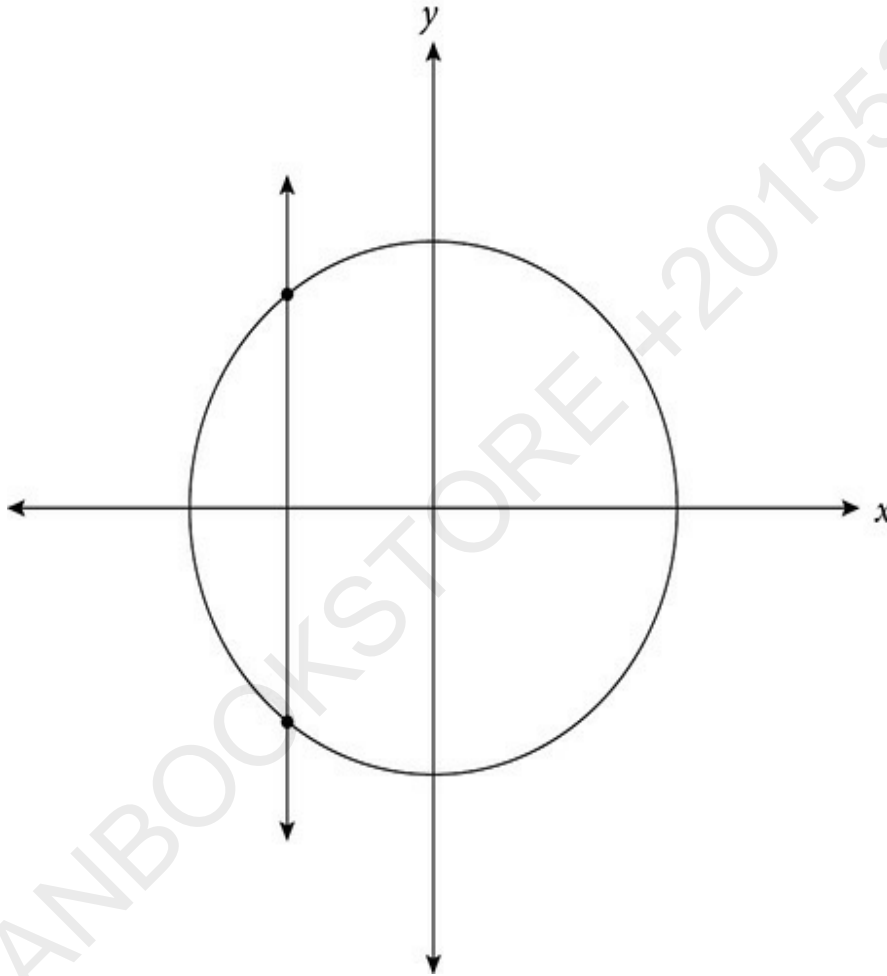
You can immediately see that for one value of x there are two corresponding values of y . Let $x = 4$, for example. $y = \pm \sqrt{(5 + 4)}$, so $y = \pm 3$.

$y^2 = 5 + x$ is not a function. Answer

Example:

Is $x^2 + y^2 = 4$ a function?

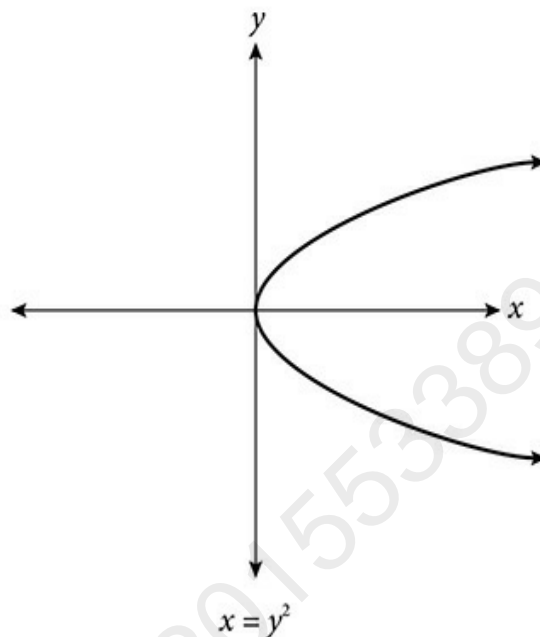
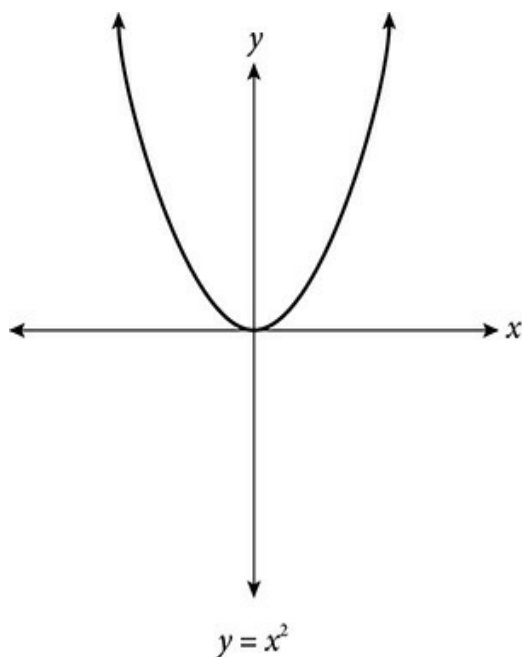
Let's test this one using the Vertical Line Test. You may be able to recognize that the graph is a circle, since $x^2 + y^2 = 4$ fits into the standard form of the equation of a circle: $(x - h)^2 + (y - k)^2 = r^2$. Its center is $(0, 0)$ and its radius is 2. A vertical line intersects the circle at two points as shown:



The equation does not pass the Vertical Line Test.

$x^2 + y^2 = 4$ is not a function. Answer

An equation that is not a function is called a relation. By definition, a *relation* is any set of ordered pairs. A function, therefore, is simply a type of relation consisting of ordered pairs with different x -coordinates. The two parabolas below show the difference between a relation and a function.



The graph of $y = x^2$ passes the Vertical Line Test, so it is a function. The graph of $x = y^2$ does not pass the Vertical Line Test (when $x = 4$, $y = 2$ or $y = -2$), so it is a relation.

Example:

Which of the following is NOT a function?

- (A) $\{(x, y): y = x\}$
- (B) $\{(x, y): x = |y|\}$
- (C) $\{(x, y): x^2 = y\}$
- (D) $\{(x, y): y = \sqrt{x}\}$
- (E) $\{(x, y): y = x^3\}$

Each answer is written in set notation and represents a set of ordered pairs. Answer A, for example, is read “the set of all ordered pairs (x, y) such that $y = x$.”

- (A) $y = x$ is a linear equation and passes the Vertical Line Test.
- (B) $x = |y|$ is a “V-shaped” graph that is concave left and does not pass the vertical line. When $x = 5$, for example, $y = \pm 5$.
- (C) $x^2 = y$ is a parabola whose vertex is the origin and is concave up.

(D) $y = \sqrt{x}$ is a curve starting at the origin and moving upward in the positive x direction. Since it is a square root equation, it is not defined when x is negative. The range, y , is restricted to positive values because y equals a square root.

(E) $y = x^3$ is a cubic curve that passes through the origin. The domain is the set of all real numbers, and each x has one y associated with it.

B is not a function. Answer

Graphing Functions

The most straightforward way to graph a function is to create an xy -table and find the coordinates of points for which the function is true. Start with the intercepts; set $y = 0$ and solve for x , then set $x = 0$ and solve for y . Knowing the graphs of a few common functions may save you time. Be familiar with graphing the following functions:

- Identity function $f(x) = x$
- Constant function $f(x) = c$
- Absolute value function $f(x) = |x|$
- Squaring function $f(x) = x^2$
- Square root function $f(x) = \sqrt{x}$
- Cubing function $f(x) = x^3$

Composition of Functions

Two functions can be combined to form a *composition*. $f(g(x))$ is read as “the composition of f with g .” Let $f(x) = x^2$ and $g(x) = x + 1$, and evaluate the following:

$$f(g(x)) = f(x + 1) = (x + 1)^2$$

$$g(f(x)) = g(x^2) = x^2 + 1$$

$$f(f(x)) = f(x^2) = x^4$$

$$f(g(2)) = f(3) = 9$$

Notice that $f(g(x))$ is not equivalent to $g(f(x))$ in this example.

Example:

Given $g(x) = x + 3$, $h(x) = 9 - x^2$, and $g(h(x)) = 8$, find x .

$$g(h(x)) = g(9 - x^2) = 9 - x^2 + 3 = 12 - x^2$$

You're given that $g(h(x)) = 8$, so set $12 - x^2$ equal to 8 and solve for x .

$$12 - x^2 = 8$$

$$4 = x^2$$

$$x = \pm 2 \quad \text{Answer}$$

Example:

Given $f(x) = x^{\frac{2}{3}}$ and $g(x) = x^6$, find $f(g(8x))$.

Start by finding $g(x)$.

$$g(8x) = (8x)^6$$

Now find $f((8x)^6)$.

$$\begin{aligned} f((8x)^6) &= [(8x)^6]^{\frac{2}{3}} = 8^{6\left(\frac{2}{3}\right)} x^{6\left(\frac{2}{3}\right)} \\ &= 8x^4 \\ &= 4,096 x^4 \quad \text{Answer} \end{aligned}$$

Identity, Zero, and Constant Functions

The *identity function* is the function for which $y = x$. $f(x) = x$. Its graph is a diagonal line passing through the origin whose slope is 1.

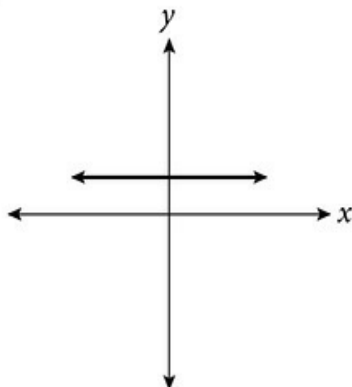
The *zero function* is the function that assigns 0 to every x . $f(x) = 0$. Its graph is the horizontal line in which $y = 0$, otherwise known as the x -axis.

A *constant function* is any function that assigns a constant value c to every x . $f(x) = c$. Its graph is a horizontal line, $y = c$, whose y -intercept is the point $(0, c)$.

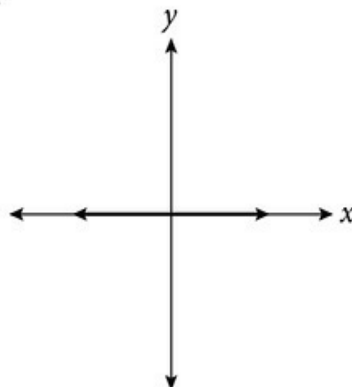
Example:

Which of the following graphs represent(s) a constant function?

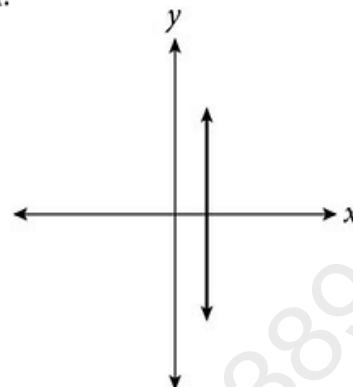
I.



II.



III.



- (A) I only
- (B) II only
- (C) III only
- (D) I and II
- (E) II and III

III is not a function since it does not pass the Vertical Line Test. I represents the constant function, $y = c$. II represents the zero function, $y = 0$, but the zero function is also a constant function.

D is the correct answer. Answer

Determining the Maximum or Minimum

The graph of a function can be increasing, decreasing, or constant for intervals of its domain. A function is:

- **Decreasing.** If the y -value decreases from left to right
- **Increasing.** If the y -value increases from left to right
- **Constant.** If the y -value remains unchanged from left to right

Take parabola $y = x^2$, for example. The graph decreases when $x < 0$ and increases when $x > 0$. The *maximum* or *minimum* value of a function is often the point where the function changes behavior from decreasing to increasing.

The Maximum or Minimum of a Quadratic Function

Let $f(x) = ax^2 + bx + c$ where $a \neq 0$.

If $a < 0$, the parabola is concave down and has a maximum value.

If $a > 0$, the parabola is concave up and has a minimum value.

The maximum or minimum is $f(x)$ when $x = \frac{-b}{2a}$. In other words, it's the y-value of the parabola's vertex. It's $f\left(\frac{-b}{2a}\right)$.

Example:

Find the maximum value of the function $f(x) = -4x^2 + 3x + 1$.

Notice a is less than zero, so the function does, in fact, have a maximum value. Solve for the x -coordinate of the function's vertex:

$$x = \frac{-b}{2a} = \frac{-3}{2(-4)} = \frac{3}{8}$$

Now find the value of the function when $x = \frac{3}{8}$.

$$\begin{aligned} f\left(\frac{3}{8}\right) &= -4\left(\frac{3}{8}\right)^2 + 3\left(\frac{3}{8}\right) + 1 \\ &= -\frac{9}{16} + \frac{9}{8} + 1 = \frac{-9 + 18 + 16}{16} = \frac{25}{16} \end{aligned}$$

The maximum value is $\frac{25}{16}$ or ≈ 1.563 Answer

Example:

A gardener has 60 feet of fencing to enclose a rectangular garden. Find the greatest possible area of the garden.

Let w = the width of the garden and l = the length of the garden.

$$2w + 2l = 60$$

$$w + l = 30$$

Write l in terms of w : $l = 30 - w$. The area of the garden is $A = lw$, so substitute $30 - w$ in for l .

$$A = (30 - w)w$$

Notice that this results in a quadratic equation.

$$A = 30w - w^2 \text{ where } a = -1 \text{ and } b = 30.$$

The maximum value occurs when $w = \frac{-b}{2a} = \frac{-30}{2(-1)} = 15$.

When $w = 15$, $l = 30 - 15 = 15$, so the area is $15(15)$ or 225 square feet.

225 square feet Answer

The Roots of a Quadratic Function

The *roots*, or *solutions*, of a quadratic equation are values of the variable that satisfy the equation. For example, factoring $x^2 + 2x - 15 = 0$ results in $(x + 5)(x - 3) = 0$. $x = -5$ and $x = 3$ are the roots of the equation. When dealing with functions, the roots are the x -values that result when $f(x) = 0$. You can also think of the roots as the x -intercepts of the graph of a function or the *zeros* of the function.

If you're given the roots of a quadratic equation, you can use them to determine the equation itself. A quadratic equation can be thought of as:

$$a[x^2 - (\text{sum of the roots})x + (\text{product of the roots})] = 0$$

Example:

The product of the roots of a quadratic equation is -14 and their sum is -3 . Find a quadratic equation whose roots have the given product and sum.

Since the sum of the two roots is -3 and the product is -14 , substitute these values into the equation.

$$a[x^2 - (\text{sum of the roots})x + (\text{product of the roots})] = 0$$

$$a[x^2 - (-3)x + (-14)] = 0$$

Setting a equal to 1 results in one possible answer:

$$x^2 + 3x - 14 = 0 \quad \text{Answer}$$

Example:

Find a quadratic equation with integral coefficients having roots 6 and 2.

The sum of the roots is 8 and their product is 12.

$$a[x^2 - (\text{sum of the roots})x + (\text{product of the roots})] = 0$$

$$a(x^2 - 8x + 12) = 0$$

Again, setting a equal to 1 results in one possible answer:

$$x^2 - 8x + 12 = 0 \quad \text{Answer}$$

Example:

Find a quadratic equation with integral coefficients having roots $-\sqrt{2}$ and $\sqrt{2}$.

The sum of the roots is 0 and their product is -2 .

$$a[x^2 - 0x + (-2)] = 0$$

Setting a equal to 1 results in one possible answer:

$$x^2 - 2 = 0 \quad \text{Answer}$$

Inverse Functions

The *inverse of a function* is denoted by f^{-1} and satisfies the compositions $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. f^{-1} is not necessarily a function (i.e., it doesn't have to pass the Vertical Line Test). A function has an inverse, however, if it passes the *Horizontal Line Test*. This means that if every horizontal line intersects the graph of a function in *at most* one point, then an inverse of the function exists.

If (a, b) is a point on the graph of f , then (b, a) is a point on the graph of f^{-1} . In other words, the domain of f equals the range of f^{-1} , and the range of f equals the domain of f^{-1} . Because of this property, the graphs of f and f^{-1} are reflections of each other with respect to the line $y = x$. Some examples of functions and their inverse are as follows:

$f(x) = x + 2$	$f^{-1}(x) = x - 2$
$g(x) = \frac{(x + 3)}{2}$	$g^{-1}(x) = 2x - 3$
$h(x) = x^3 - 1$	$h^{-1}(x) = \sqrt[3]{x + 1}$
$f(x) = x^2$	no inverse
$g(x) = \frac{1}{x}$	$g^{-1}(x) = \frac{1}{x}$
$h(x) = \sqrt[4]{x}$	$h^{-1}(x) = x^2$

On your graphing calculator, graph $y_1 =$ one of the given functions above, $y_2 =$ its inverse function, and $y_3 = x$ to see that the function and its inverse are reflections, or “mirror images,” of each other over the line $y = x$. Notice that the squaring function, $f(x) = x^2$, does not have an inverse, since it doesn't pass the Horizontal Line Test. The square root function, $h(x) = \sqrt[4]{x}$, does, however, have an inverse because its domain is restricted to $x \geq 0$.

To algebraically solve for the inverse of a function, interchange y and x and solve for the new y -value. This is the inverse function, if an inverse exists.

For example, take $f(x) = \frac{x+4}{2}$. Replacing $f(x)$ by y results in $y = \frac{x+4}{2}$.

Now interchange x and y and solve for y :

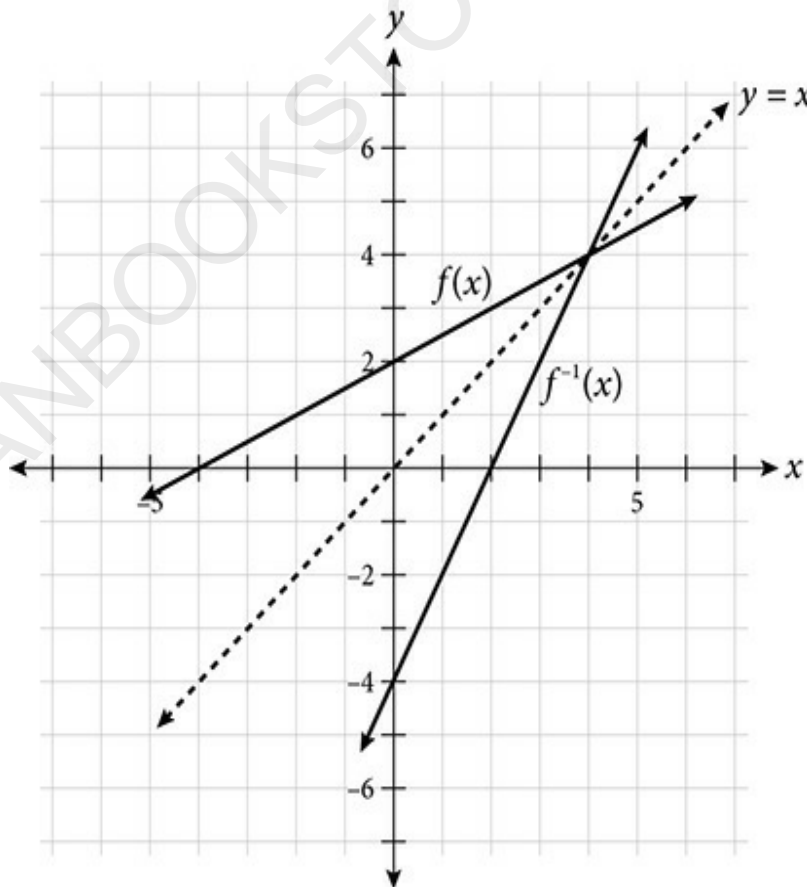
$$x = \frac{y+4}{2}$$

$$2x = y + 4$$

$$2x - 4 = y$$

$$f^{-1} = 2x - 4$$

Check your answer by graphing f and f^{-1} to see that they are reflections over the line $y = x$.



Example:

If $f(x) = \frac{x+6}{3}$ and f^{-1} is the inverse function of f , what is $f^{-1}(-1)$?

$$f(x) = \frac{x+6}{3}$$

$$y = \frac{x+6}{3}$$

$$x = \frac{y+6}{3}$$

$$3x = y + 6$$

$$3x - 6 = y$$

$$f^{-1} = 3x - 6$$

$$f^{-1}(-1) = 3(-1) - 6 = -9 \quad \text{Answer}$$

Example:

If $f(x) = 4x - 1$, then what is $f^{-1}(x)$?

Interchange the x - and y -values in the function $f(x) = 4x - 1$ and solve for y .

$$y = 4x - 1$$

$$x = 4y - 1$$

$$x + 1 = 4y$$

$$f(x)^{-1} = \frac{x+1}{4} \quad \text{Answer}$$

Example:

If $g(x) = x^4$, then what is $g^{-1}(x)$?

$$y = x^4$$

$$x = y^4$$

$\sqrt[4]{x} = y$. There is not a one-to-one correspondence between x - and y -values.

No inverse Answer

Rational Functions

A *rational function* is defined by a rational (i.e., fractional) expression in one variable and can be written as $f(x) = \frac{p(x)}{q(x)}$. Unlike the linear and quadratic

functions that have been discussed thus far in this chapter, rational functions are not necessarily continuous. They contain a break in the graph at the point where the denominator equals zero. (Limits of rational functions are discussed in the Number and Operations chapter.) The graph of a rational function f has *vertical asymptotes* at the zeros of the denominator $q(x)$.

Example:

Find the domain of $f(x) = \frac{x^2 - 4}{x^2 - 4x}$.

The domain is the set of x -values for which the function is defined, so $x^2 - 4x$ cannot equal zero.

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

The domain is all real numbers except $x = 0$ and $x = 4$. Answer

Note that the graph of $f(x) = \frac{x^2 - 4}{x^2 - 4x}$ has vertical asymptotes at $x = 0$ and $x = 4$.

Many graphing calculators do not handle asymptotes well, so you may get a graph that appears to be continuous with “zigzags” at $x = 0$ and $x = 4$. The graph should be discontinuous at $x = 0$ and $x = 4$, since the function is undefined at these domain values. On a TI graphing calculator, try changing the Mode to Dot, instead of Connected, to better view the graph.

Example:

Find the zeros of $f(x) = \frac{x^3 - 3x}{x^2 - 9}$.

The zeros of the function occur when $f(x) = 0$, or, in other words, at the x -intercepts of the graph.

$$0 = \frac{x^3 - 3x}{x^2 - 9}$$

$$0 = x^3 - 3x \quad (x \neq \pm 3)$$

$$0 = x(x^2 - 3)$$

$$0 = x \quad \text{or} \quad x^2 = 3$$

$$x = \pm\sqrt{3},$$

Example:

What are the equations of the asymptotes of $f(x) = \frac{x^3}{x^2 - 1}$?

Vertical asymptotes occur at the zeros of the denominator.

$$x^2 - 1 = 0$$

$$x = \pm\sqrt{1} = \pm 1$$

Note there are no horizontal asymptotes, since the degree of the numerator is greater than the degree of the denominator.

$x = 1$ and $x = -1$ Answer

Higher-Degree Polynomial Functions

A *polynomial function* of x with degree n is given by:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x^1 + a_0$$

where n is a nonnegative number, the coefficients of the x terms are real numbers, and $a_n \neq 0$.

A first-degree polynomial function is a *linear function*: $f(x) = ax + b$ ($a \neq 0$).

A second-degree polynomial function is a *quadratic function*: $f(x) = ax^2 + bx + c$ ($a \neq 0$).

The graphs of polynomial functions share the following properties:

1. They are continuous.
2. They have rounded curves.
3. If n (the highest exponent) is odd and $a_n > 0$, the graph falls to the left and rises to the right.
4. If n (the highest exponent) is odd and $a_n < 0$, the graph rises to the left and falls to the right.
5. If n (the highest exponent) is even and $a_n > 0$, the graph rises to the left and right.
6. If n (the highest exponent) is even and $a_n < 0$, the graph falls to the left and right.

Properties 3 through 6 describe what is known as the *Leading Coefficient Test*, which describes the right and left behavior of the graphs of functions.

Example:

Determine the right and left behavior of the graph of $f(x) = -x^3 + 7x$.

The degree of the function is odd ($n = 3$) and its leading coefficient is -1 .

The graph rises to the left and falls to the right. Answer

The *zeros of a polynomial function* are the x -values when $f(x) = 0$. A function of degree n has at most n real zeros.

Example:

Find the real zeros of $f(x) = x^3 - 2x^2 - 8x$.

Since $f(x)$ is a third-degree function, it can have, at most, three zeros.

$$x^3 - 2x^2 - 8x = 0$$

$$x(x^2 - 2x - 8) = 0$$

$$x(x - 4)(x + 2) = 0$$

$$x = 0, x = 4, \text{ and } x = -2 \quad \text{Answer}$$

Example:

Find the real zeros of $f(x) = -x^4 + 2x^3 - x^2$.

Since $f(x)$ is a fourth-degree function, it can have, at most, four zeros.

$$-x^4 + 2x^3 - x^2 = 0$$

$$-x^2(x^2 - 2x + 1) = 0$$

$$-x^2(x - 1)^2 = 0$$

$$x = 0 \text{ and } x = 1 \quad \text{Answer}$$

Note that both $x = 0$ and $x = 1$ are *repeated zeros*. Since the exponent on the factors is even, the graph touches the x -axis at these points, but it does not cross the x -axis. (If the exponent on the factors were odd, the graph

would cross the x -axis at the repeated zeros.)

Long division and synthetic division are also useful in factoring and finding the zeros of polynomial functions.

Example:

Divide $2x^3 - 9x^2 + 7x + 6$ by $x - 2$.

Let's divide using *synthetic division*:

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 7 & 6 \\ & & 4 & -10 & -6 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

The rightmost digit, 0, is the remainder. This means that $x - 2$ divides evenly into $2x^3 - 9x^2 + 7x$.

The quotient is $2x^2 - 5x - 3$. **Answer**

The *Division Algorithm* states that:

$$f(x) = d(x)q(x) + r(x)$$

where $d(x)$ is the *divisor*, $q(x)$ is the *quotient*, and $r(x)$ is the *remainder*.

Applying this algorithm to the last example results in:

$$f(x) = (x - 2)(2x^2 - 5x - 3)$$

You can further factor $2x^2 - 5x - 3$ to get

$$f(x) = (x - 2)(2x + 1)(x - 3)$$

Using synthetic division helps to factor polynomials that are not special products and easily factorable.

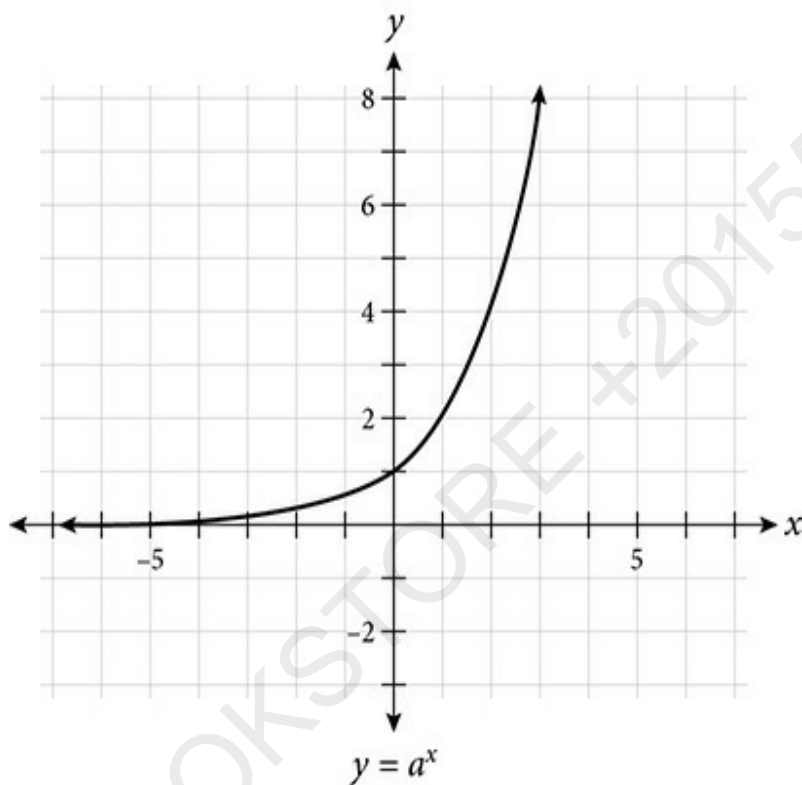
Exponential Functions

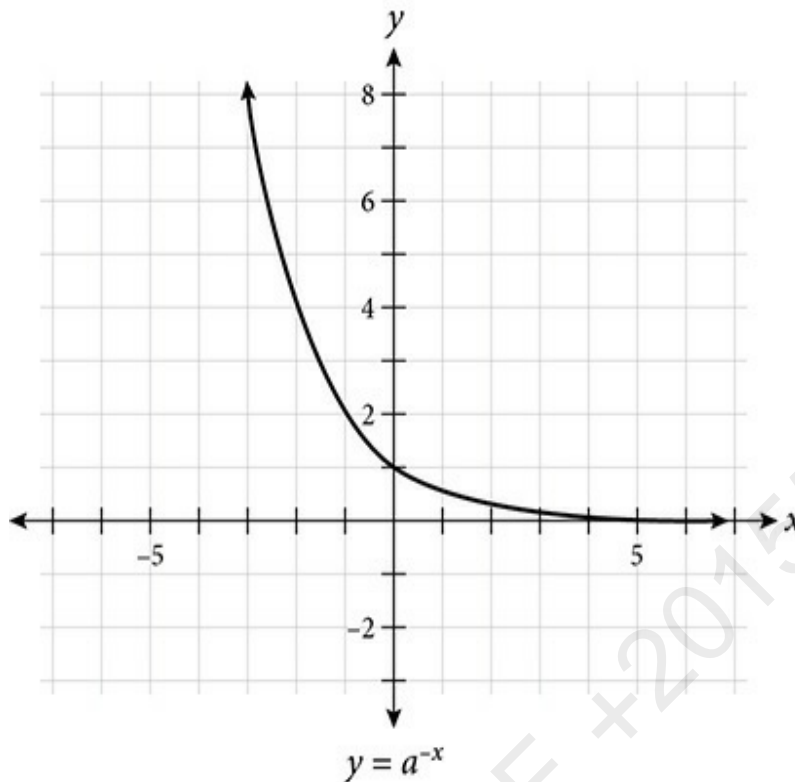
An exponential function f with base a is given by:

$$f(x) = a^x$$

where x is a real number, $a > 0$, and $a \neq 1$.

The graphs of $y = a^x$ and $y = a^{-x}$ are as follows:





Notice that both have a y-intercept of 1 and a horizontal asymptote at $y = 0$. The graph of $y = a^{-x}$ is a reflection of the graph of $y = ax$ over the y-axis.

Example:

Use the properties of exponents to determine if the functions $f(x) = 27(3^{-x})$ and $g(x) = \left(\frac{1}{3}\right)^{x-3}$ are the same.

Review the properties of exponents given in the Algebra chapter if you're unsure of how to solve this problem.

$$f(x) = 27(3^{-x}) = 3^3(3^{-x}) = 3^{3-x}$$

$$g(x) = \left(\frac{1}{3}\right)^{x-3} = (3^{-1})^{x-3} = 3^{-x+3} = 3^{3-x}$$

$f(x)$ and $g(x)$ are the same functions. Answer

Example:

If $f(x) = 2x$ and $g(x) = 3x$, then determine when $3x < 2x$.

Both functions have a y -intercept of 0. Graph the functions on your graphing calculator (use the \wedge button to raise 2 and 3 to the x power) to compare the curves. When $x > 0$, $3x > 2x$.

$3x < 2x$ when $x < 0$. Answer

REVIEW QUESTIONS

1. What is $f^{-1}(x)$ if $f(x) = \frac{1}{2}x - 2$?

- (A) $x - 4$
- (B) $x + 2$
- (C) $x + 4$
- (D) $2x + 2$
- (E) $2x + 4$

2. The chart in [Figure 1](#) shows an exponential function. What is the next y -value?

x	y
-2	.5
-1	1.5
0	4.5
1	13.5
2	?

Figure 1

- (A) 14.5
 - (B) 25.5
 - (C) 27.0
 - (D) 40.5
 - (E) 53.5
3. What is the maximum value of the function $f(x) = -2x^2 - 3x + 4$?
- (A) 5.13
 - (B) 3.38
 - (C) 2.25
 - (D) 1.13
 - (E) 0.56
4. If $f(x) = x^2 + 3$ and $g(x) = x - 2$, what is $f(g(x))$?
- (A) $x^2 - 4x + 7$
 - (B) $x^2 - 2x + 3$
 - (C) $x^2 - 2x + 4$
 - (D) $2x^2 + 3$
 - (E) $2x^2 + 7$
5. What is $3x^2 - 6x + 13$ divided by $x + 2$?
- (A) $3x + 6$
 - (B) $3x - 12$
 - (C) $3x - 6$
 - (D) $3x^2 - 12x + 37$
 - (E) $3x^2 - 6x + 24$

ANSWERS AND EXPLANATIONS

1. (E) Set $f(x) = y$. $y = \frac{1}{2}x - 2$

Switch x and y . $x = \frac{1}{2}y - 2$

Solve for y . $\frac{1}{2}y = x + 2$

$$y = 2(x + 2)$$

$$2x + 4 = y$$

$$f^{-1}(x) = 2x + 4$$

2. (D) Find the pattern. Each y -value is the product of the preceding y -value and 3. The value of y when $x = 1$ is 13.5, so the value of y when $x = 2$ will be $13.5 \times 3 = 40.5$.

3. (A) Solve for the x -coordinate of the vertex.

$$x = \frac{-b}{2a} = \frac{3}{2(-2)} = \frac{3}{-4}$$

Now find the value of the function when $x = \frac{-3}{4}$.

$$f\left(\frac{-3}{4}\right) = -2\left(\left(\frac{-3}{4}\right)^2\right) - 3\left(\frac{-3}{4}\right) + 4$$

$$f\left(\frac{-3}{4}\right) = -2\left(\frac{9}{16}\right) + \frac{9}{4} + 4$$

$$f\left(\frac{-3}{4}\right) = -\frac{9}{8} + \frac{9}{4} + 4 = \frac{41}{8} = 5.125 = 5.13$$

4. (A) $g(x) = x - 2$, so $f(g(x)) = f(x - 2)$

$$\begin{aligned} f(x - 2) &= (x - 2)^2 + 3 \\ &= x^2 - 4x + 4 + 3 \\ &= x^2 - 4x + 7 \end{aligned}$$

5. (B) Use synthetic division.

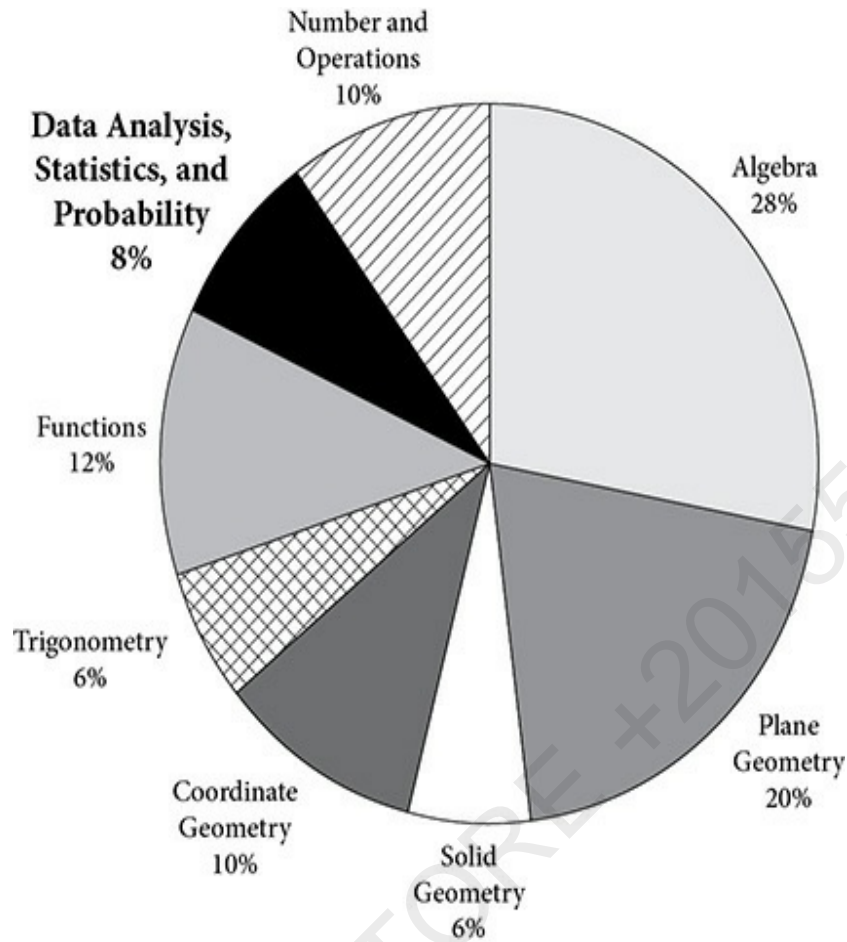
$$\begin{array}{r|rrr} -2 & 3 & -6 & 13 \\ & & -6 & 24 \\ \hline & 3 & -12 & 37 \end{array}$$

37 is the remainder. The quotient is $3x - 12$.



Data Analysis, Statistics, and Probability

This chapter provides a review of elementary statistics. Six to ten percent of the Level 1 test questions relate specifically to data analysis, statistics, and probability. The statistics on the Level 1 test focuses on measures of central tendency, graphs and plots, linear regression and basic probability, and counting problems. Permutations, combinations, and standard deviation are included on the Level 2 test. The pie chart below shows approximately how much of the Level 1 test is related to data analysis, statistics, and probability:



The following topics are covered in this chapter:

1. Counting Problems
2. Probability
3. Mean, Median, Mode
4. Data Interpretation

Counting Problems

The *Fundamental Counting Principle* states that if one action can be done in a ways, and for each of these a second action can be done in b ways, the number of ways the two actions can be done in order is $a \times b$.

For example, if an automobile manufacturer produces 4 different models of cars and each one is available in 5 different colors, there are

$$4 \times 5 = 20$$

20 different combinations of car model and color that can be created.

Mutually exclusive events are events that cannot occur at the same time. For example, when you roll a die, you either roll a 1, 2, 3, 4, 5, or 6. 1, 2, 3, 4, 5, and 6 are mutually exclusive events. When you flip a coin, you get either heads or tails. Heads and tails are also mutually exclusive. If the possibilities being counted are mutually exclusive, then the total number of possibilities is the *sum* of the number of possibilities in each group.

Example:

How many positive integers between 0 and 100 can be created using the digits 1, 2, 3, 4, and 5?

Consider all the possible 1-digit and 2-digit integers that can be created using 1, 2, 3, 4, or 5. These are *mutually exclusive events*.

1-digit integers: 5 possibilities for the ones digit

2-digit integers: (5 possibilities for the tens digit)(5 possibilities for the ones digit)

Since the possibilities being counted are mutually exclusive, find the sum of the number of possibilities in each group.

$$5 + 5(5) = 30$$

30 integers can be created. Answer

Example:

John's high school offers 6 math courses, 4 English courses, and 3 science courses to students in his grade level. How many schedules are possible if John chooses a course in each subject?

Use the Fundamental Counting Principle to get

$$6 \times 4 \times 3 = 72 \text{ schedules}$$

72 schedules Answer

Example:

In how many different ways can a 15-question true or false quiz be answered? Assume every question must be answered.

There are 15 “events” and each one has two possible outcomes. Using the Fundamental Counting Principle, you get

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{15} = 32,768$$

32,768 possible combinations of answers Answer

Now, try the previous example assuming that you can leave questions blank. The number of true/false questions doesn't change, but now you have 3 possible outcomes for each question—true, false, or blank. The answer becomes 3^{15} or 14,348,907 possible combinations.

Example:

Rosemarie wears a uniform to work. As part of her uniform, she can wear one of 3 pairs of pants, one of 4 shirts, and one of 2 hats. How many pant-shirt-hat combinations are possible?

Again, use the Fundamental Counting Principle to get

$$3 \times 4 \times 2 = 24$$

24 possible pant-shirt-hat combinations Answer

Example:

A certain state makes license plates consisting of 6 symbols (letters and numbers), and it is a requirement that the letters always come before the numbers. How many license plates of 6 symbols (letters and numbers) can be made using at least one letter in each?

There are 6 mutually exclusive events possible—license plates with 1, 2, 3, 4, 5, or 6 letters. Recognize that there are 26 possibilities for choosing a letter and 10 possibilities for choosing a number (the digits 0 through 9)

and its okay to use a number or letter more than once.

$$1 \text{ Letter and 5 Numbers:} \quad 26 \times 10 \times 10 \times 10 \times 10 \times 10 = 2,600,000$$

$$2 \text{ Letters and 4 Numbers:} \quad 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000$$

$$3 \text{ Letters and 3 Numbers:} \quad 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

$$4 \text{ Letters and 2 Numbers:} \quad 26 \times 26 \times 26 \times 26 \times 10 \times 10 = 45,697,600$$

$$5 \text{ Letters and 1 Number:} \quad 26 \times 26 \times 26 \times 26 \times 26 \times 10 = 118,813,760$$

$$6 \text{ Letters:} \quad 26 \times 26 \times 26 \times 26 \times 26 \times 26 = 308,915,776$$

The total number of possible license plates is the sum of the 6 mutually exclusive events:

$$2,600,000 + 6,760,000 + 17,576,000 + 45,697,600 + 118,813,760 + 308,915,776 = 500,363,136$$

500,363,136 possible license plates Answer

Example:

How many different seating arrangements can be made for 5 students in a row of 5 desks?

This problem is different from the previous ones, since order matters. Choose the student to sit in the first desk. There are 5 possibilities for choosing that student. Once one student is seated, he or she cannot be chosen again, so there are only 4 possible students who can be chosen to sit in the second desk. Using that logic, there are 3 possible students to choose for the third desk, 2 for the fourth, and 1 for the fifth. The problem can be solved using multiplication:

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

Notice that $5 \times 4 \times 3 \times 2 \times 1 = 5!$. You can use a *factorial* to solve problems in which order matters. A *permutation* is an ordered arrangement of elements. The number of permutations of n objects is $n!$. Hence there are $5!$ possible permutations in this problem.

120 seating arrangements Answer

Probability

An experiment is an occurrence in which you do not necessarily get the same results when it is repeated under similar conditions. The *sample space* is the set of all possible outcomes of an experiment. When you toss a coin, the possible outcomes are heads, H, or tails, T. The sample space of a coin toss is written as {H, T}. The sample space for rolling a die, for example, is {1, 2, 3, 4, 5, 6}. An *event* is a set of outcomes and is a subset of the sample space.

If all outcomes are equally likely, the *probability* that an event, E , occurs is

$$P(E) = \frac{\text{the number of possible outcomes of } E}{\text{the total number of possible outcomes}}$$

Example:

Two dice are rolled. Find the probability that the sum of the two numbers is less than 4.

When the two dice are rolled there are

$6 \times 6 = 36$ total possible outcomes

The sum of the two dice must, however, be less than 4. If the first die is a 1, the second could be 1 or 2. If the first die is 2, the second die could be a 1. If the first die is a 3, 4, 5, or 6, there are no possibilities that the second roll will result in a sum of less than 4. The possible outcomes are therefore

{(1, 1), (1, 2), (2, 1)}

The probability is $\frac{\text{the number of possible outcomes}}{\text{the total number of possible outcomes}} = \frac{3}{36} = \frac{1}{12}$.

$\frac{1}{12}$ Answer

Example:

There are 12 pieces of colored chalk in a package—3 white, 3 yellow, 3 orange, and 3 green. If two pieces are selected at random, find the probability that both will be yellow.

The probability of choosing the first yellow piece of chalk is

$$P(Y_1) = \frac{3}{12} = \frac{1}{4}$$

Once one piece is chosen, there are only 11 pieces left in the package. The probability of choosing the second piece of yellow chalk is

$$P(Y_2) = \frac{2}{11}$$

The answer is therefore

$$\frac{1}{4} \left(\frac{2}{11} \right) = \frac{2}{44} = \frac{1}{22}$$

$$\frac{1}{22} \quad \text{Answer}$$

Notice that in the previous problem the first piece of chalk is *not replaced* before the second is drawn. This decreases the total number of outcomes to 11 when the second piece of chalk is selected.

Example:

The probability of passing this week's math test is 70 percent, and the probability of passing this week's English test is 80 percent. What is the probability of failing both tests?

To get the probability that an event will NOT occur, subtract the probability that the event *will* occur from 1.

The probability of not passing this week's math test is $1 - 70\% = 30\%$.

The probability of not passing this week's English test is $1 - 80\% = 20\%$.

Notice that these are *independent events*, meaning that passing the math test is not dependent on how you do on the English test and vice versa. Multiply the probability of not passing both tests to get

$$30\% (20\%) = 6\%$$

6% Answer

Mean, Median, Mode

Three different statistics are commonly used to measure central tendency. They are

Mean. The average of the numbers

Median. The middle number (when the data is ordered)

Mode. The number that occurs the most

The *mean* is calculated by finding the sum of all the terms and dividing by the total number of terms. After the data is ordered, the *median* is simply the middle value of an odd number of terms or the average of the two middle values for an even number of terms. The *mode* is the most frequent value. It is possible for data to have more than one mode.

Example:

The ages of the starting players on a high school soccer team are as follows:

14, 15, 15, 16, 16, 17, 17, 17, 17, 18

Find the mean, median, and mode of the data.

1. Mean. Calculate the mean by finding the sum of all the ages and dividing that by the number of ages in question.

The mean is $\frac{14 + 15 + 15 + 16 + 16 + 16 + 17 + 17 + 17 + 17 + 18}{11}$

$$= \frac{178}{11} \approx 16.18$$

2. Median. The sixth term of 11 total terms is the middle number.

The median is 16.

3. Mode. 17 occurs 4 times in the given data.

The mode is 17.

Mean \approx 16.18, Median = 16, Mode = 17 Answer

Example:

Sarah has test scores of 65, 78, 81, 82, and 90. What must she score on her sixth test to maintain an average score of 80?

Let x be Sarah's sixth test score. Since you know the average, or mean, of the data, set up an equation equal to 80 and solve for x .

$$80 = \frac{65 + 78 + 81 + 82 + 90 + x}{6}$$

$$80 = \frac{396 + x}{6}$$

$$480 = 396 + x$$

$$84 = x$$

Sarah must score 84 on her sixth test. Answer

Example:

Find the median and mode of the following distribution:

0, 0, 1, 4, 6, 8, 8, 9

The *mode* is the number that occurs the most. Since both 0 and 8 occur twice, each number is a mode.

The *median* is the middle number. Since there's an *even* number of terms in the given distribution, the median cannot be one of the given terms. Instead, find the median by adding the fourth and fifth terms and dividing by 2. (In essence, you're finding the average of the two terms, since the middle occurs between them.) The median is

$$\frac{4 + 6}{2} = 5$$

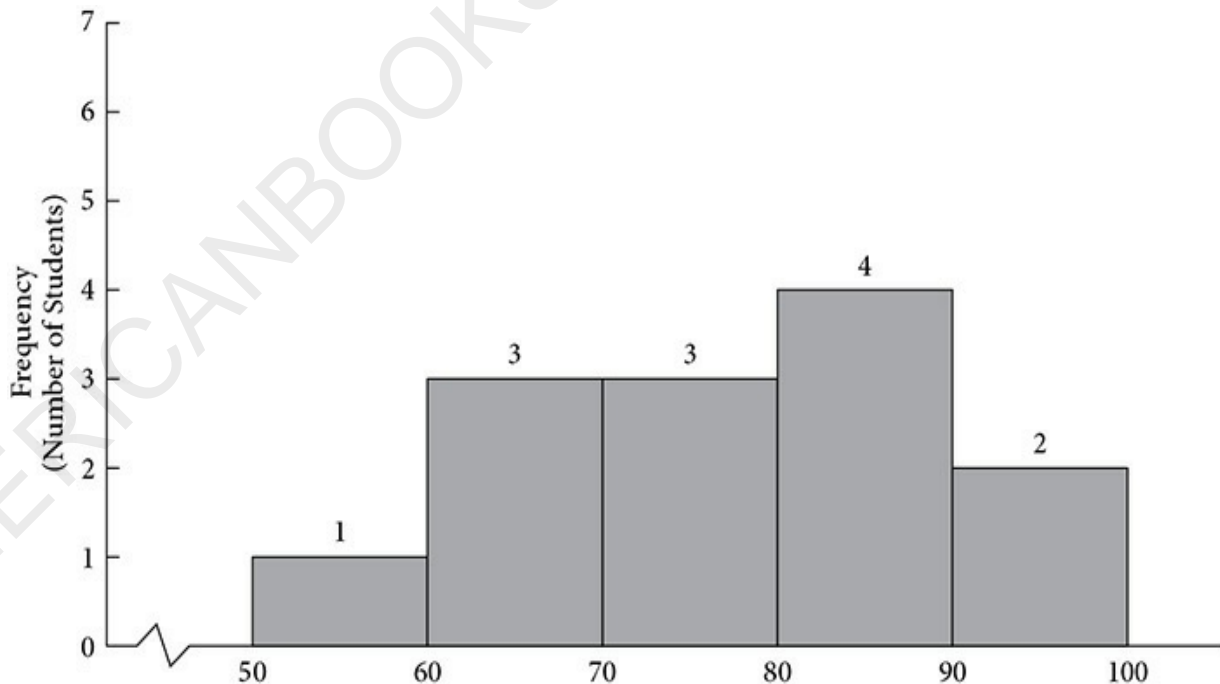
The mode is 0 and 8, and the median is 5. Answer

Data Interpretation

Data interpretation problems involve reading data from a histogram, pie chart, frequency distribution, bar graph, or other type of data display.

Example:

The following histogram shows students' scores on a given test:



Each of the following is true EXCEPT:

- (A) The interval 80–89 contains the most scores.
- (B) The median score is in the interval 70–79.
- (C) 9 students scored 70 or above.
- (D) 13 students took the test.
- (E) One student got 100 percent.

The most number of students scored in the interval between 80 and 89, since it is the highest bar in the histogram, making answer A correct. There's not enough information to find the exact median of the data, but you can see that the middle term, the seventh term, falls in the interval between 70 and 79. Answer B is therefore also correct. Nine ($3 + 4 + 2$) students scored higher than 70 and 13 students ($1 + 3 + 3 + 4 + 2$) students took the test, so answers C and D are correct. That leaves E. On quickly looking at the graph, you may mistakenly think that the highest score on the test was 100 percent. A histogram represents *intervals* of data, however. Two students scored in the highest interval, but you don't know enough information to conclude that a student received 100 percent.

E is the correct answer. Answer

Example:

The frequency distribution of the average daily temperatures in June is shown below. Determine the mean temperature. Round your answer to the nearest integer.

TEMPERATURE	FREQUENCY
59	2
60	5
61	6
62	6
63	5
64	3
65	2
66	1

The *mean* is the sum of all the temperatures divided by the total number of days in question. Adding up the frequencies results in:

$$2 + 5 + 6 + 6 + 5 + 3 + 2 + 1 = 30 \text{ days}$$

June actually has 30 days, so this number makes sense. Now find the sum of the 30 given temperatures and divide by 30.

$$\frac{59(2) + 60(5) + 61(6) + 62(6) + 63(5) + 64(3) + 65(2) + 66(1)}{30}$$

$$= \frac{1,859}{30}$$

$$\approx 61.97$$

62 degrees Answer

REVIEW QUESTIONS

Questions 1 to 3 refer to the following record of student grades on a biology test.

SCORE	FREQUENCY
95	2
90	4
85	6
80	5
75	4
70	1

1. What is the sum of the median and the mode of the data set?

- (A) 85
- (B) 165
- (C) 170
- (D) 172.5
- (E) 175

2. What is the mean score?

- (A) 80
- (B) 82.50
- (C) 83.18
- (D) 84.16
- (E) 85

3. What percent of the students scored above 80?

- (A) 77%
- (B) 55%

- (C) 50%
- (D) 43%
- (E) 36%
4. The first 5 runners to cross the finish line in a race receive ribbons. If 30 runners are in the race, how many different groups of ribbon winners are possible?
- (A) 120
- (B) 3,125
- (C) 142,506
- (D) 6,375,600
- (E) 17,100,720
5. On Tuesday, there is a 60 percent probability of rain. On Wednesday, there is a 30 percent probability of rain. What is the probability that it will rain on Wednesday, but not on Tuesday?
- (A) 12%
- (B) 18%
- (C) 28%
- (D) 42%
- (E) 45%

ANSWERS AND EXPLANATIONS

1. (C) The median is the middle value. Add up the number of scores: $2 + 4 + 6 + 5 + 4 + 1 = 22$. The middle value will be the average of values 11 and 12 since 22 is even. Both value 11 and value 12 are 85, so the median is 85. The mode is the most frequently occurring value. The score with the highest frequency is 85, so the mode is 85. The sum of the median and mode is $85 + 85 = 170$.
2. (C) To find the mean, add all the values and divide by the number of values. Multiply the score by its frequency, and then add those totals.

$95 \times 2 = 190$, $90 \times 4 = 360$, $85 \times 6 = 510$, $80 \times 5 = 400$, $75 \times 4 = 300$, $70 \times 1 = 70$.

$190 + 360 + 510 + 400 + 300 + 70 = 1,830$. $1,830 \div 22 \text{ values} = 83.18$.

3. **(B)** First, count how many students scored above 80. $2 + 4 + 6 = 12$ students.

Divide by the total number of students. $12 \div 22 = .5454 = 55\%$

4. **(C)** There are 30 possibilities for the first position, then 29, then 28, then 27, then 26. Multiply the number of options for each position together.

$30 \times 29 \times 28 \times 27 \times 26 = 17,100,720$. Now divide by the number of ways to arrange 5 people: $5 \times 4 \times 3 \times 2 \times 1 = 120$. $17,100,720$ divided by $120 = 142,506$.

5. **(A)** The probability that it will rain on Wednesday is 30 percent. The probability that it will NOT rain on Tuesday is 40 percent. For both to happen, multiply the individual probabilities together.

$.30 \times .40 = .12 = 12\%$

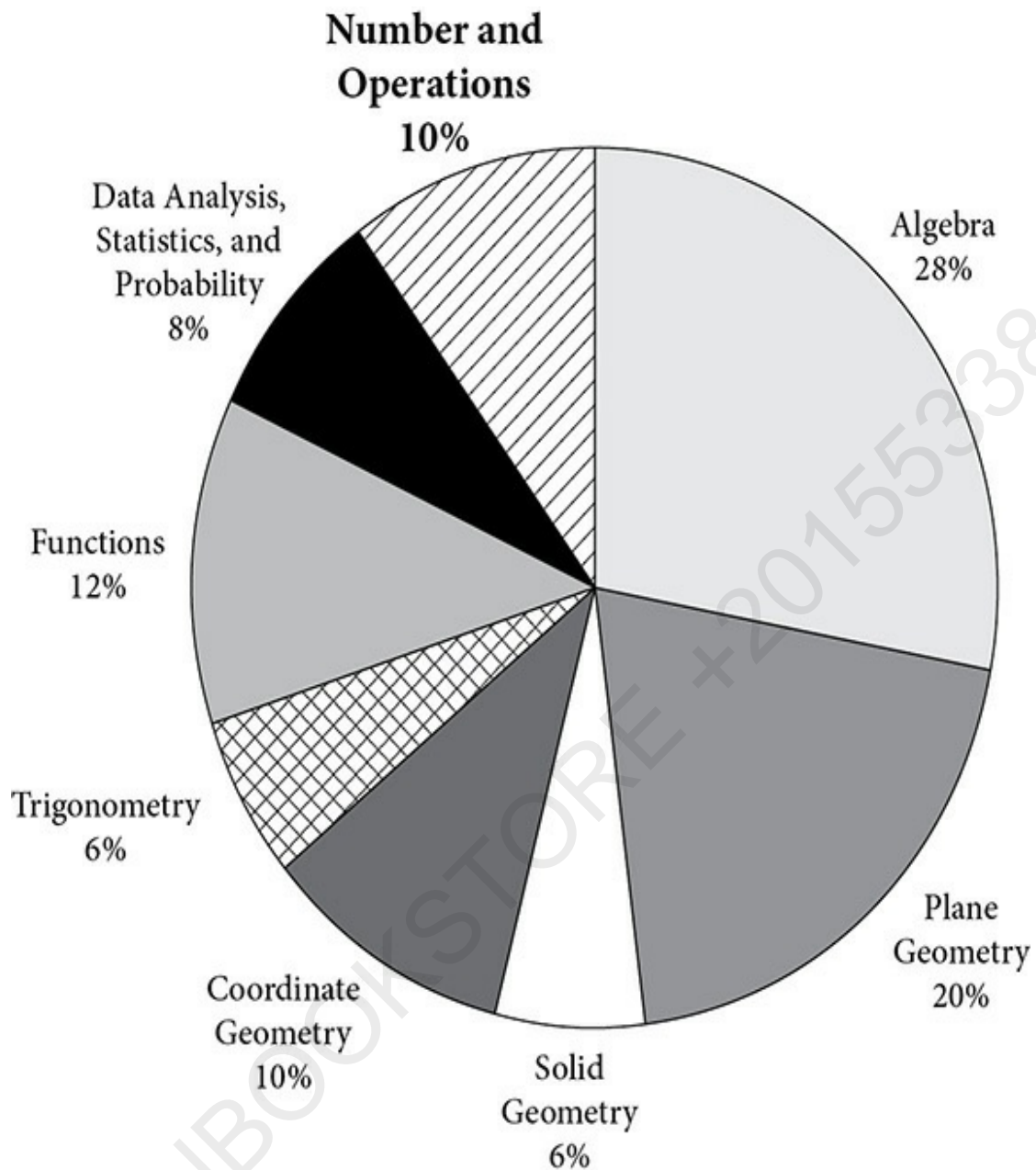


Number and Operations

This chapter provides a review of number sense and operations. 10 to 14 percent of the Level 1 test questions relate to Number and Operations (formerly called Miscellaneous topics). These questions cover items outside the realm of the algebra, geometry, trigonometry, and statistics found on the rest of the test. Number and Operations questions on the Level 1 test tend to focus on logic, number theory, and sequences.

The College Board claims that students are not expected to have studied every topic. You should expect some test questions, especially those under the Number and Operations category, to seem unfamiliar to you. In the practice tests at the end of this book, miscellaneous problems involving complex numbers, exponential functions, and logarithms are categorized under Number and Operations because they do not fit under any other category.

The pie chart below shows approximately how much of the Level 1 test is related to Number and Operations:



The following topics are covered in this chapter:

1. Invented Operations
2. "In Terms Of" Problems
3. Sequences
 - a. Arithmetic Sequences
 - b. Geometric Sequences
4. Logic
5. Number Theory

Invented Operations

“Invented operations” appear often in standardized tests. This type of question introduces a new symbol that represents a made-up mathematical operation. Think back to learning what the symbols $+$, $-$, \times , \div , and $\sqrt{\quad}$ represent. When presented with a plus sign, $+$, you learned to add the number on the left of the symbol to the number on the right. Don’t panic when the symbol presented in one of these problems looks unfamiliar. It should, since the symbol is made up for the problem.

Example:

The operation \boxplus is defined for all real numbers x and y by the equation:

$$x \boxplus y = 4x + \frac{y^2}{2}. \text{ If } n \boxplus 4 = 12, \text{ find the value of } n.$$

Remember that \boxplus should NOT look familiar to you. Don’t panic because you don’t know what the symbol means! The question defines the symbol for you:

$$x \boxplus y = 4x + \frac{y^2}{2}$$

When $x = n$ and $y = 4$, the result is 12. Substitute n and 4 into the equation for x and y , and solve for n .

$$n \boxplus 4 = 4n + \frac{4^2}{2} = 12$$

$$4n + 8 = 12$$

$$4n = 4$$

$$n = 1 \quad \text{Answer}$$

Example:

The operation ∇ is defined for all real numbers a and b by the equation:

$a \nabla b = |a|^b$. If $-6 \nabla n = 216$, find the value of n .

Just like the previous example, the question defines the symbol ∇ for you:

$$a \nabla b = |a|^b$$

When $a = -6$ and $b = n$, the result is 216. Substitute -6 and n into the equation for a and b , and solve for n .

$$-6 \nabla n = |-6|^n = 216$$

$$6^n = 216$$

6 raised up to some power n equals 216. You know that $6^1 = 6$ and $6^2 = 36$. You may or may not know off the top of your head what 6^3 equals. $6 \times 6 \times 6 = 36 \times 6 = 216$. (On a graphing calculator, type $6 \wedge 3$.)

$$n = 3 \quad \text{Answer}$$

“In Terms Of” Problems

Sometimes on the Level 1 test, you may be faced with one equation with more than one unknown variable. You don't have enough information to solve for an unknown, but you can solve for one variable *in terms of* the other variables.

Example:

Given $2x - 3y = x - 6y + 9$, solve for y in terms of x .

Your goal is to isolate y on one side of the equation. This will give you the solution for y *in terms of* x .

$$2x - 3y = x - 6y + 9$$

$$2x + 3y = x + 9$$

$$3y = -x + 9$$

$$y = \frac{-x + 9}{3} \quad \text{Answer}$$

Example:

Given $a + b = \frac{x}{d} + \frac{x}{c}$, find the value of x in terms of a , b , c , and d .

Again, your goal is to isolate x on one side of the equation. Start by multiplying both sides by the LCD, cd .

$$a + b = \frac{x}{d} + \frac{x}{c}$$

$$cd(a + b) = cx + dx$$

$$cd(a + b) = x(c + d)$$

$$\frac{cd(a + b)}{(c + d)} = x \quad \text{Answer}$$

Sequences

A *sequence* is a set of numbers listed in a certain order. You can also think of a sequence as a function whose domain is the set of consecutive positive integers. a_n is a term in the sequence and n is the number of the term. *Finite sequences* have a limited number of terms, while *infinite sequences* continue indefinitely. Some examples of infinite sequences are as follows:

$$1, 2, 3, 4, 5, \dots n, \dots$$

$$2, 4, 6, 8, 10, \dots 2n, \dots$$

$$3, 9, 27, 81, 243, \dots 3^n, \dots$$

$$1, 4, 9, 16, 25, \dots n^2, \dots$$

$$1, 4, 7, 10, 13, \dots 3n - 2, \dots$$

In general, infinite sequences are in the form: $a_1, a_2, a_3, a_4, a_5, a_n, \dots$

Arithmetic Sequences

If the difference between consecutive terms in a sequence is a constant, the sequence is an *arithmetic sequence*. The constant difference between terms is called the *common difference* and is represented by d . In a given arithmetic sequence $a_1, a_2, a_3, a_4, a_5, \dots$, the difference between any two consecutive terms must equal d : $a_2 - a_1 = d$, $a_3 - a_2 = d$, $a_4 - a_3 = d$, and so on.

The n th term of an arithmetic sequence can be found using the equation $a_n = dn + c$, where d is the common difference, n is the number of the term, and c is a constant. Alternately, the equation $a_n = a_1 + (n - 1)d$ (where a_1 is the first term of the sequence) also represents the n th term of an arithmetic sequence.

The sum of a finite arithmetic sequence is: $S_n = \frac{n}{2}(a_1 + a_n)$.

Example:

Find the n th term of the arithmetic sequence 6, 10, 14, 18, 22, \dots

Since the sequence is arithmetic, there is a common difference between consecutive terms. $d = 10 - 6 = 4$. The n th term must be in the form:

$$a_n = dn + c$$

$$a_n = 4n + c$$

Because $a_1 = 6$, you can write an expression for the first term and solve for c .

$$6 = 4(1) + c$$

$$2 = c$$

The formula for the n th term is $an = 4n + 2$. Answer

Notice the last example can also be solved using the equation $an = a_1 + (n - 1)d$. Substituting $a_1 = 6$ and $d = 4$ results in:

$$an = 6 + (n - 1)4$$

$$an = 6 + 4n - 4 = 4n + 2$$

Example:

Find the sum of the terms in the sequence 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30.

Of course, you could simply add these 15 terms to get an answer. A better way is to recognize that this is, in fact, an arithmetic sequence. $n = 15$, $a_1 = 2$, and $a_n = 30$. Substitute these values into the formula for the sum of a finite arithmetic sequence to get

$$S_n = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24 + 26 + 28 + 30$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{15}{2}(2 + 30)$$

$$S_n = \frac{15}{2}(32)$$

$$S_n = 15(16) = 240$$

$$240 \quad \text{Answer}$$

Example:

Find the sum of the integers from 1 to 200.

The integers from 1 to 200 form an arithmetic sequence having 200 terms. $n = 200$, $a_1 = 1$, and $a_n = 200$. Substitute these values into the formula for the sum of a finite arithmetic sequence to get

$$S_n = 1 + 2 + 3 + 4 + 5 + \dots + 200$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{200}{2}(1 + 200)$$

$$S_n = 100(201) = 20,100$$

$$20,100 \quad \text{Answer}$$

Geometric Sequences

If the ratio between consecutive terms in a sequence is a constant, the sequence is a *geometric sequence*. The constant ratio between terms is called the *common ratio* and is represented by r . In a given geometric sequence $a_1, a_2, a_3, a_4, a_5, \dots$, the ratio between any two consecutive terms must equal r :

$$\frac{a_2}{a_1} = r, \frac{a_3}{a_2} = r, \frac{a_4}{a_3} = r, \dots, \text{ and so on, and } r \text{ cannot equal zero.}$$

The n th term of a geometric sequence can be found using the equation $a_n = a_1 r^{n-1}$, where r is the common ratio, n is the number of the term, and a_1 is the first term of the sequence. Using this formula, every geometric sequence can be written as follows:

$$a_1, a_2, a_3, a_4, a_5, \dots, a_n, \dots \quad \text{or} \quad a_1, a_1 r^1, a_1 r^2, a_1 r^3, a_1 r^4, \dots, a_1 r^{n-1}, \dots$$

The sum of a finite geometric sequence is

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right) \quad \text{where } r \neq 1$$

The sum of an infinite geometric sequence is

$$S = \frac{a_1}{1 - r} \quad \text{where } -1 < r < 1$$

Example:

Find the n th term of the geometric sequence: $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \dots$

This is a geometric sequence because there is a common ratio.

$$r = \frac{\frac{3}{2}}{3} = \frac{3}{6} = \frac{1}{2}$$

Substitute the values $r = \frac{1}{2}$ and $a_1 = 3$ into the formula for the n th term to get

$$a_n = a_1 r^{n-1}$$

$$a_n = 3 \left(\frac{1}{2} \right)^{n-1} \quad \text{Answer}$$

Example:

The second term of a geometric sequence is $\frac{1}{2}$ and the fifth term is $\frac{1}{28}$.

Find the common ratio.

Start by writing the two given terms in $a_n = a_1 r^{n-1}$ form.

$$a_2 = a_1 r^1 = \frac{1}{2} \quad \text{and} \quad a_5 = a_1 r^4 = \frac{1}{128}$$

You can then write a_5 in terms of a_2 .

$$a_2 \times r \times r \times r = a_5$$

$$\frac{1}{2} \times r \times r \times r = \frac{1}{128}$$

$$\frac{1}{2} r^3 = \frac{1}{128}$$

$$r^3 = \frac{2}{128} = \frac{1}{64}$$

$$r = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$$

$$\frac{1}{4} \quad \text{Answer}$$

Example:

Find the sum of the geometric sequence: $1, \frac{7}{10}, \frac{49}{100}, \dots$

Divide the second term by the first term to determine the common ratio, $r = \frac{7}{10}$. Since this is an infinite geometric sequence, find its sum by using

the formula:

$$S = \frac{a_1}{1 - r}$$

$$S = \frac{1}{1 - \frac{7}{10}}$$

$$S = \frac{1}{\frac{3}{10}}$$

$$S = \frac{10}{3} \quad \text{Answer}$$

Logic

The logic questions on the Level 1 test have to do with conditional statements, converses, inverses, and contrapositives. A *conditional statement* is an *if-then statement* that may or may not be true. Some examples of conditional statements are as follows:

If two lines are perpendicular, then they intersect at a 90° angle.

If given two points, then there is one and only one line determined by them.

If today is Saturday, then it is the weekend.

If today is the weekend, then it is Saturday.

Notice that the first three conditional statements are true, but the fourth one is not. If today is the weekend, then it could also be Sunday. The *negation* of an if-then statement is formed by inserting the word *not* into the statement. Negating the third example above results in: "If today is not Saturday, then it is not the weekend." In general, if an if-then statement is given as "if p , then q ," then the negation of the statement is "if not p , then not q ."

Three other if-then statements are created by switching and/or negating the if and then parts of a given conditional statement.

NAME	GENERAL FORM	EXAMPLE	TRUE OR FALSE?
Conditional	If p , then q .	If $\angle Y$ measures 100° , then it is obtuse.	True
Converse	If q , then p .	If $\angle Y$ is obtuse, then it measures 100° .	False
Inverse	If not p , then not q .	If $\angle Y$ does not measure 100° , then it is not obtuse.	False
Contrapositive	If not q , then not p .	If $\angle Y$ is not obtuse, then it does not measure 100° .	True

Let's take a look at what happens when the given conditional statement is false.

NAME	GENERAL FORM	EXAMPLE	TRUE OR FALSE?
Conditional	If p , then q .	If two angles are supplementary, then they are right angles.	False
Converse	If q , then p .	If two angles are right angles, then they are supplementary.	True
Inverse	If not p , then not q .	If two angles are not supplementary, then they are not right angles.	True
Contrapositive	If not q , then not p .	If two angles are not right angles, then they are not supplementary.	False

Notice that when the given conditional statement is true, the contrapositive is also true. When the given conditional statement is false, the contrapositive is also false. This means that the contrapositive is *logically equivalent* to the conditional statement, and because of this, logic questions on the Level 1 test often ask about the contrapositive.

Example:

The statement “If John lives in Boston, then he lives in Massachusetts” is logically equivalent to which of the following?

- I. If John lives in Massachusetts, then he lives in Boston.
- II. If John does not live in Boston, then he does not live in Massachusetts.
- III. If John does not live in Massachusetts, then he does not live in Boston.

- (A) I only
- (B) II only
- (C) III only
- (D) I and II only
- (E) I, II, and III

The given statement is a conditional in the form “if p , then q ” and it is a true statement. If you recall the properties of converse, inverse, and contrapositive statements, you can immediately determine that the contrapositive will also be true. The contrapositive is in the form “if not q , then not p ” and coincides with answer III above.

If you don’t recall the properties of converse, inverse, and contrapositive statements, take a look at each statement individually. I and II are false statements, so they cannot be logically equivalent to the given statement. The only possible answer is III.

The correct answer is C. Answer

Number Theory

Number theory questions have to do with such concepts as properties of positive and negative numbers, properties of prime numbers, properties of integers, and properties of odd and even numbers. The following examples show possible number theory questions found on the Level 1 test.

Example:

If a and b are both positive, a is odd, and b is even, which of the following must be odd?

(A) $b + 2a$

(B) ab

(C) $\frac{b}{a}$

(D) ab

(E) ba

Think of the terms in the expressions as being odd or even. The table below illustrates whether the answers are odd or even and gives a numeric example for each.

ANSWER	IN WORDS	EXAMPLE	EVEN OR ODD?
$b + 2a$	even + 2(odd)	$4 + 2(3) = 10$	Even
ab	odd \times even	$3(2) = 6$	Even
$\frac{b}{a}$	even \div odd	$\frac{12}{3} = 4$ $\frac{20}{3} \approx 6.67$	Even (if it is an integer)
a^b	odd raised to an even power	$3^2 = 9$	Odd
b^a	even raised to an odd power	$2^3 = 8$	Even

The correct answer is D.

Example:

If a is positive and b is negative, which of the following must be negative?

(A) $a + b$

- (B) $a + |b|$
- (C) $a - |b|$
- (D) $|ab|$
- (E) $|a|b$

Answers B and D are always positive because of the absolute value. Answers A and C could result in a positive or a negative value, so you cannot say that the expressions *must* result in a negative value. In answer E, $|a|$ is positive and b is negative. The product of a positive number and a negative number is always negative.

The correct answer is E.

REVIEW QUESTIONS

1. What is the sum of the first 12 multiples of 3?
 - (A) 12
 - (B) 36
 - (C) 84
 - (D) 150
 - (E) 234
2. If x is positive, then y is negative, which of the following statements must be true?
 - (A) If y is positive, then x is negative.
 - (B) If y is negative, then x is positive.
 - (C) If y is positive, then x is positive.
 - (D) x and y can never both be negative.
 - (E) x and y can never be equal.
3. If $x @ y = x^2 + y$, then what is the value of $-3 @ 4$?
 - (A) -12

- (B) -5
(C) 1
(D) 7
(E) 13
4. If $\frac{1}{2}x + y^2 = 240$, what is x in terms of y ?
- (A) $120 - y$
(B) $480 - 2y^2$
(C) $\sqrt{240 - \frac{1}{2}x}$
(D) $240 - y^2$
(E) $120 - \frac{1}{2}y^2$
5. If x is an integer, which of the following must be even?
- (A) $2x + 23$
(B) $\frac{1}{2}x + 4$
(C) $x + 2$
(D) $2x + 8$
(E) $\frac{1}{2}x$

ANSWERS AND EXPLANATIONS

1. (E) The sum of an arithmetic sequence is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{12} = \frac{12}{2}(3 + 36)$$

$$S_{12} = 6(39) = 234$$

2. **(A)** Only choice A must be true, since it is the contrapositive of the original statement. Each of the other choices can be disproven with a counterexample. Let $x = -4$ and $y = -4$. The statement “if x is positive” does not apply, so no conclusion can be drawn about y . This example makes choices B, D, and E false. Choice C can also be shown as false. For example, if $x = -4$ and $y = 2$, then choice C is false.
3. **(E)** Plug -3 and 4 into the function $x @ y = x^2 + y$, where -3 is x and 4 is y .

$$(-3)^2 + 4 = 9 + 4 = 13.$$

4. **(B)** Isolate x in $\frac{1}{2}x + y^2 = 240$

$$\frac{1}{2}x = 240 - y^2$$

$$x = 2(240 - y^2)$$

$$x = 480 - 2y^2$$

5. **(D)** Choice D must be even because any integer times 2 is even, and when you add 8 to an even number, the result is even.